Representación de funciones complejas en GeoGebra a través del método de dominio coloreado

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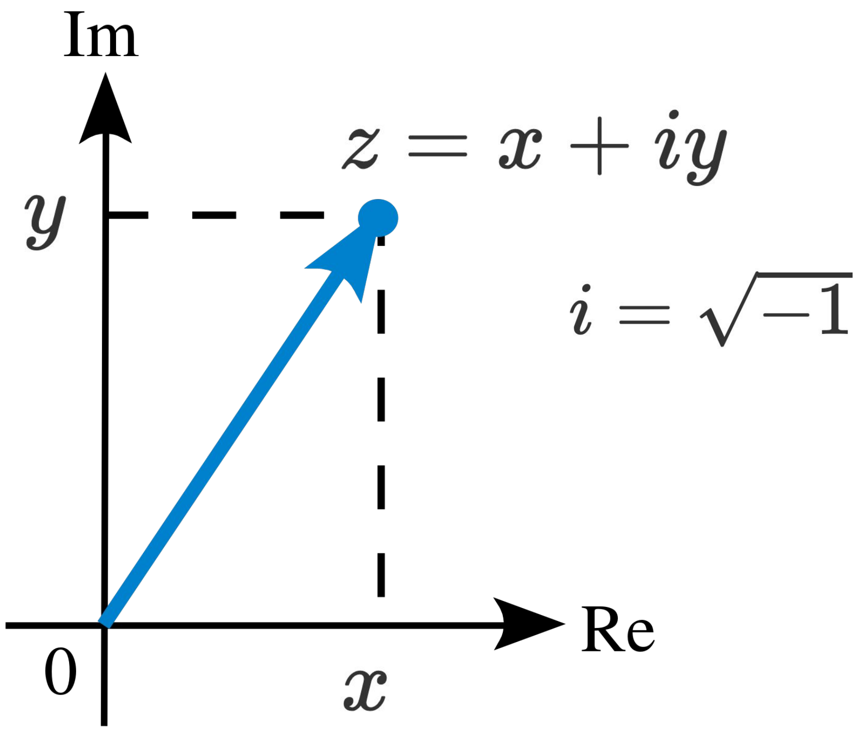
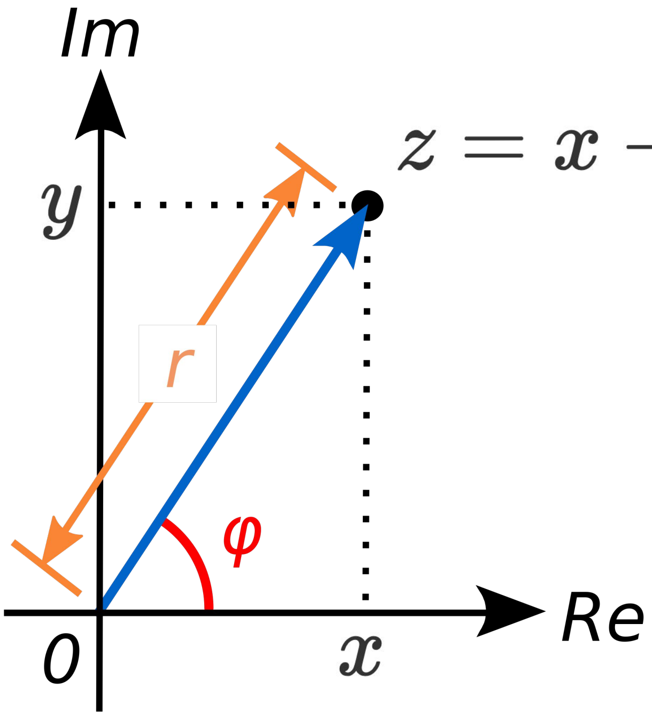
$$f : \mathbb{C} \to \mathbb{C}$$

Números complejos



Números complejos



$$z = x + iy \;= r(\cos\varphi + i\,\mathrm{sen}\varphi)$$

Módulo: $|z| = r = \sqrt{x^2 + y^2}$

Fase: $\varphi = \arg(z)$

Fase = argumento = ángulo

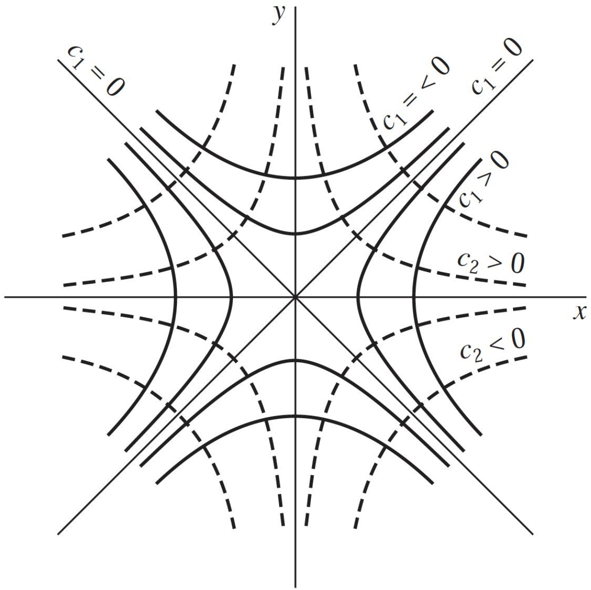
$$f : \mathbb{C} \to \mathbb{C}$$

Viven en un espacio de cuatro dimensiones.

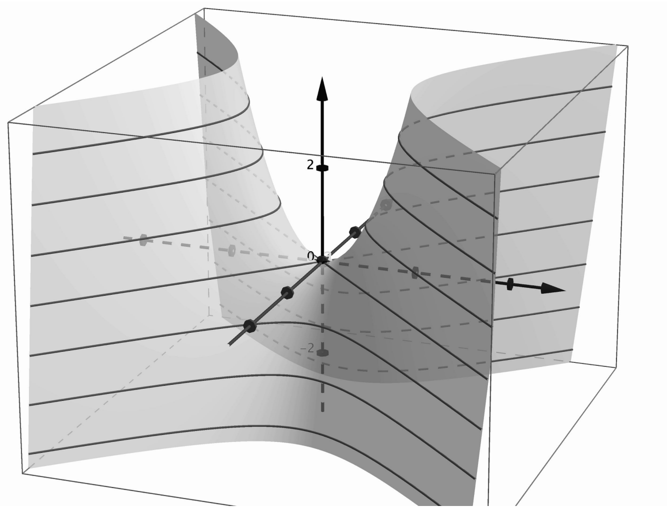
Componentes real e imaginaria

$$f(z) = \mathrm{Re}(x, y) + i\,\mathrm{Im}(x, y)$$

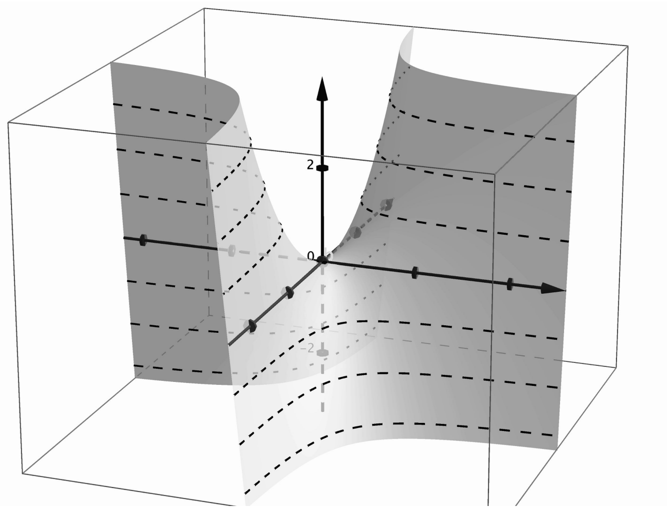
$$f(z) = z^2$$



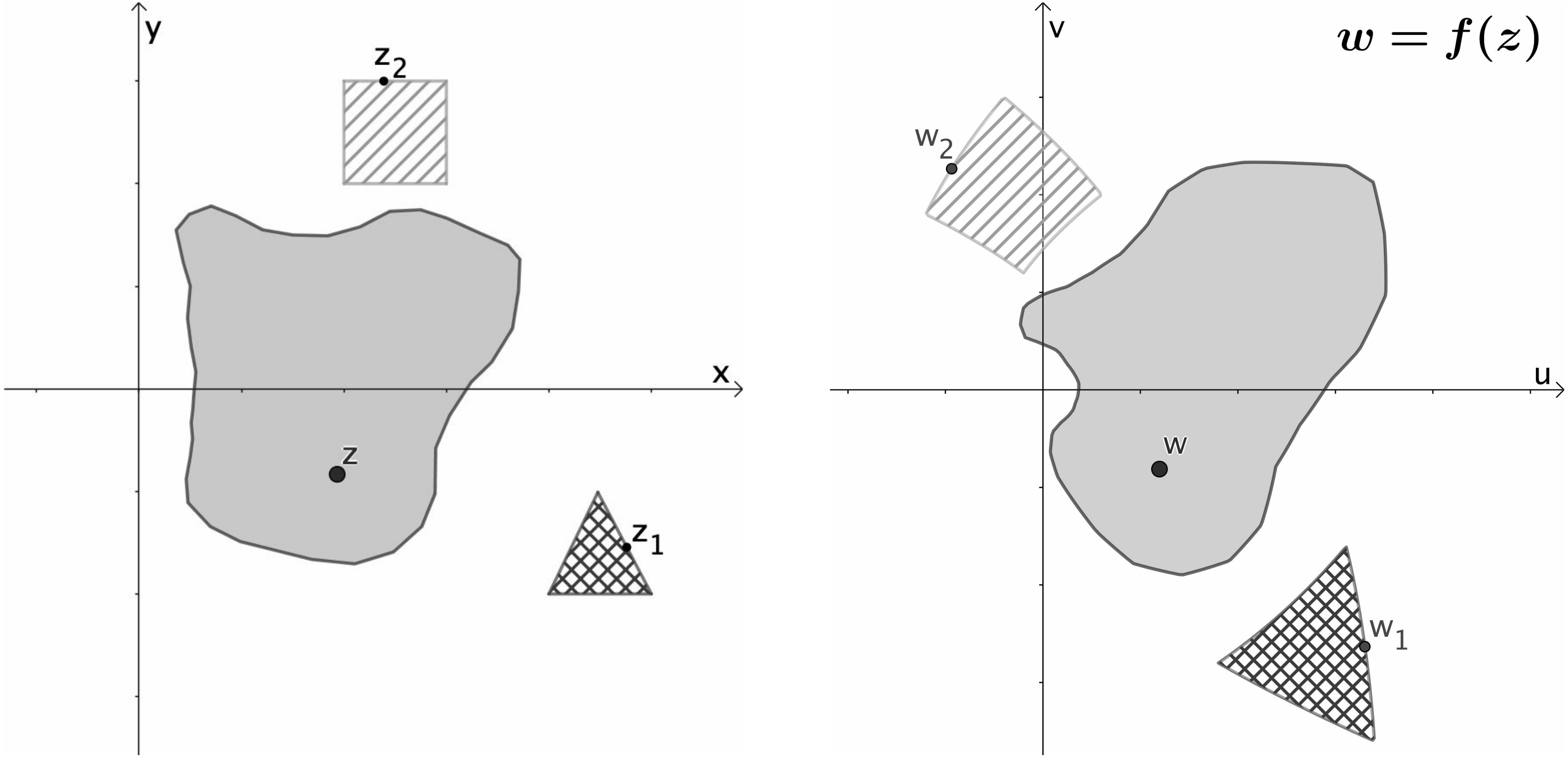
Re(x, y)



Im(x, y)

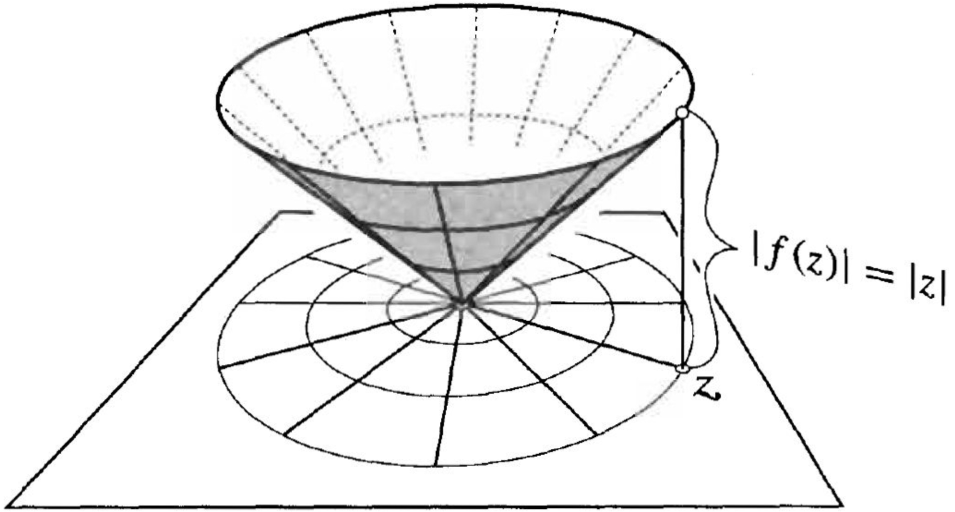


Mapeos



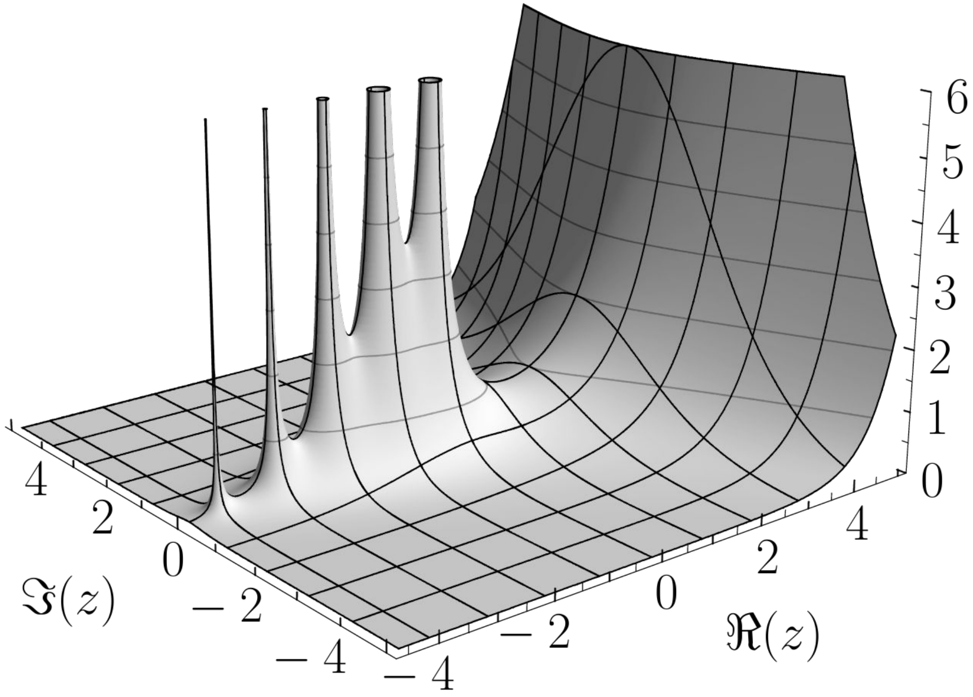
Superficies Analíticas

$$|f(z)|$$



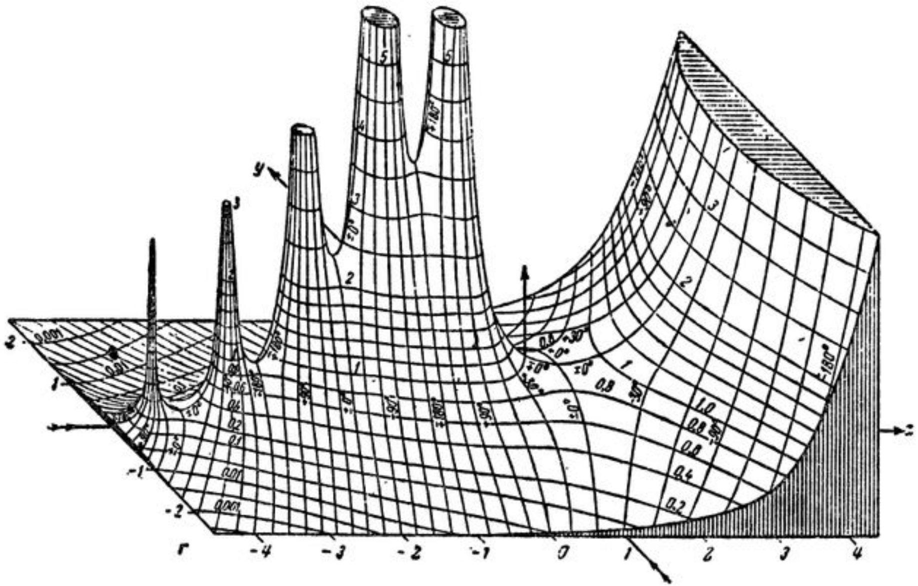
Mathematica

$|\Gamma(z)|$



Una superficie analítica histórica de 1909

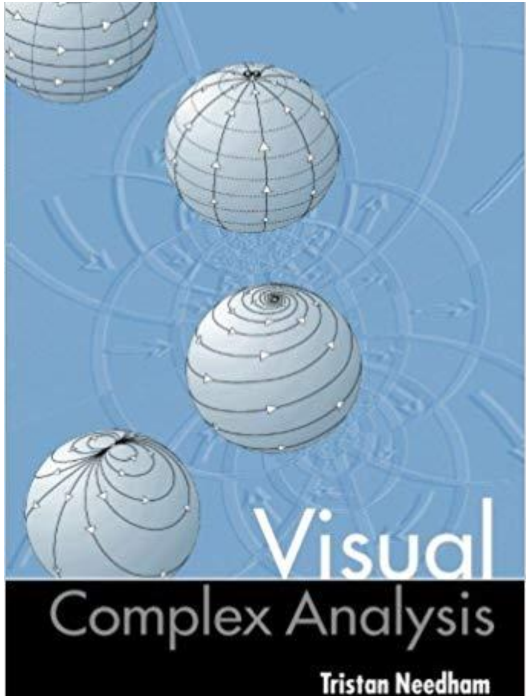
$$|\Gamma(z)|$$



Funktionentafeln mit Formeln und Kurven
por Eugene Jahnke & Fritz Emde

Dominio coloreado

- El uso de esquemas de colores, en dos dimensiones, para visualizar funciones complejas se ha usado desde finales de los 80s.
- 1998 - Reseña de Frank Farris del libro de Tristan Needham.



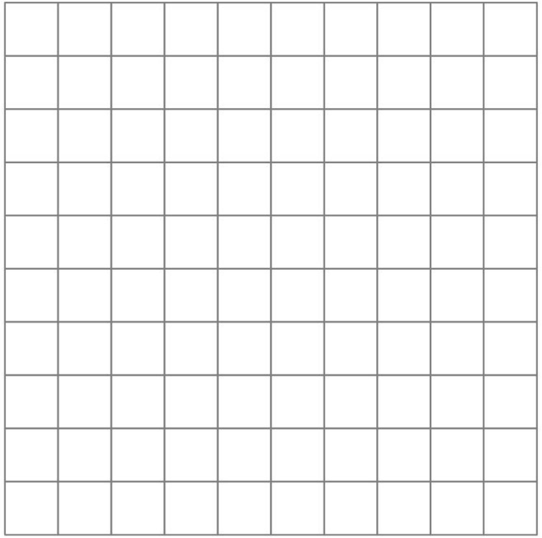
Domino coloreado

1. Asigna un color a cada punto en el plano complejo.

   HSL: Hue, Saturation & Lightness.
   HUE: Hue, Saturation & Value.

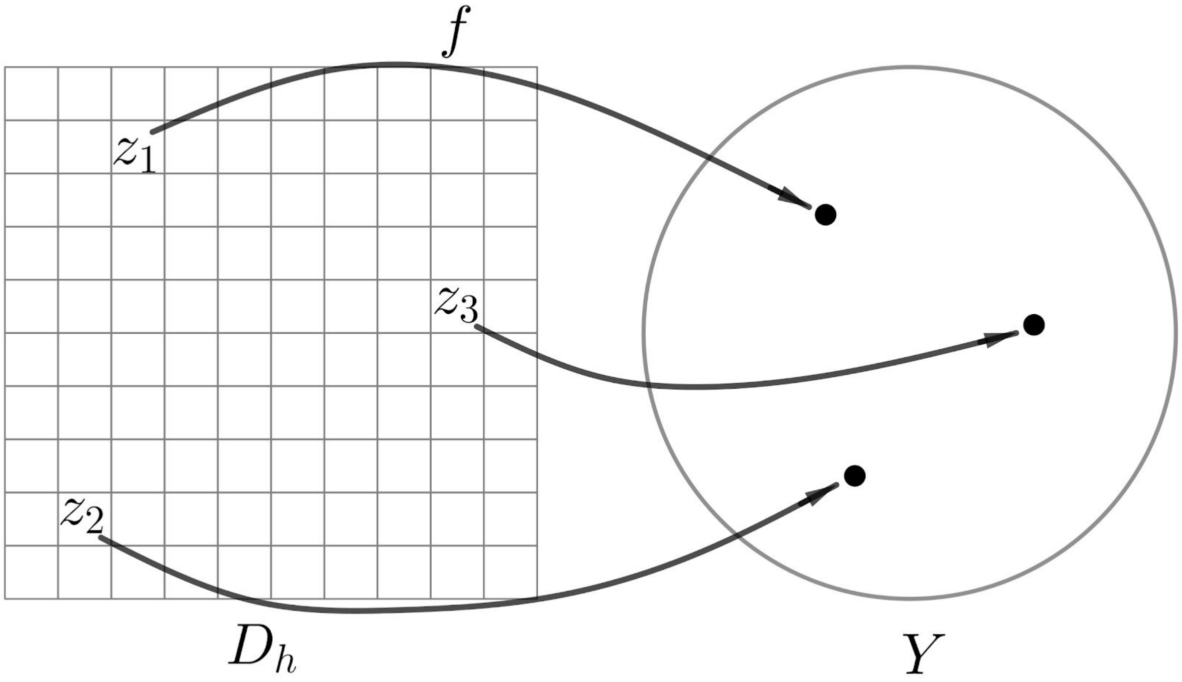
2. Colorea el dominio de f al pintar la ubicación de z en el plano con el color del valor de la función $f(z)$.

Implementación en la computadora

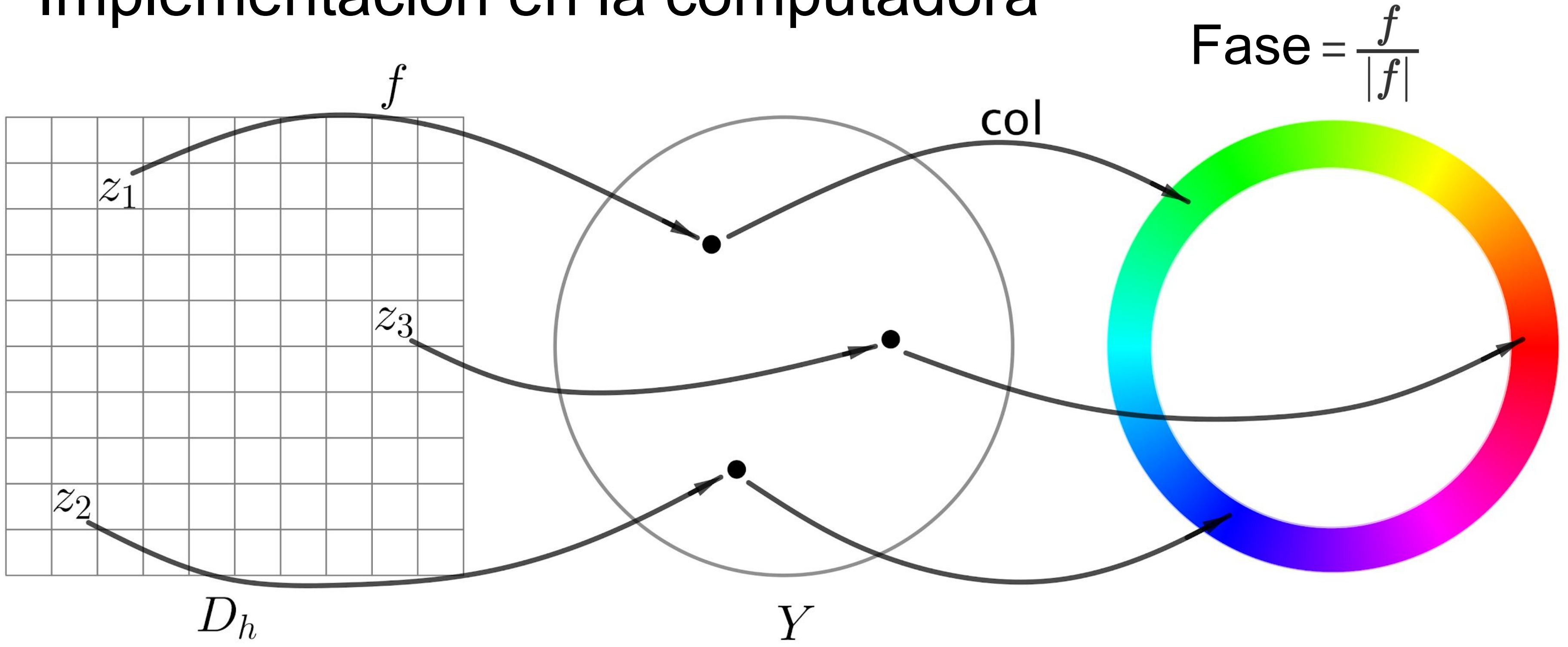


D_h

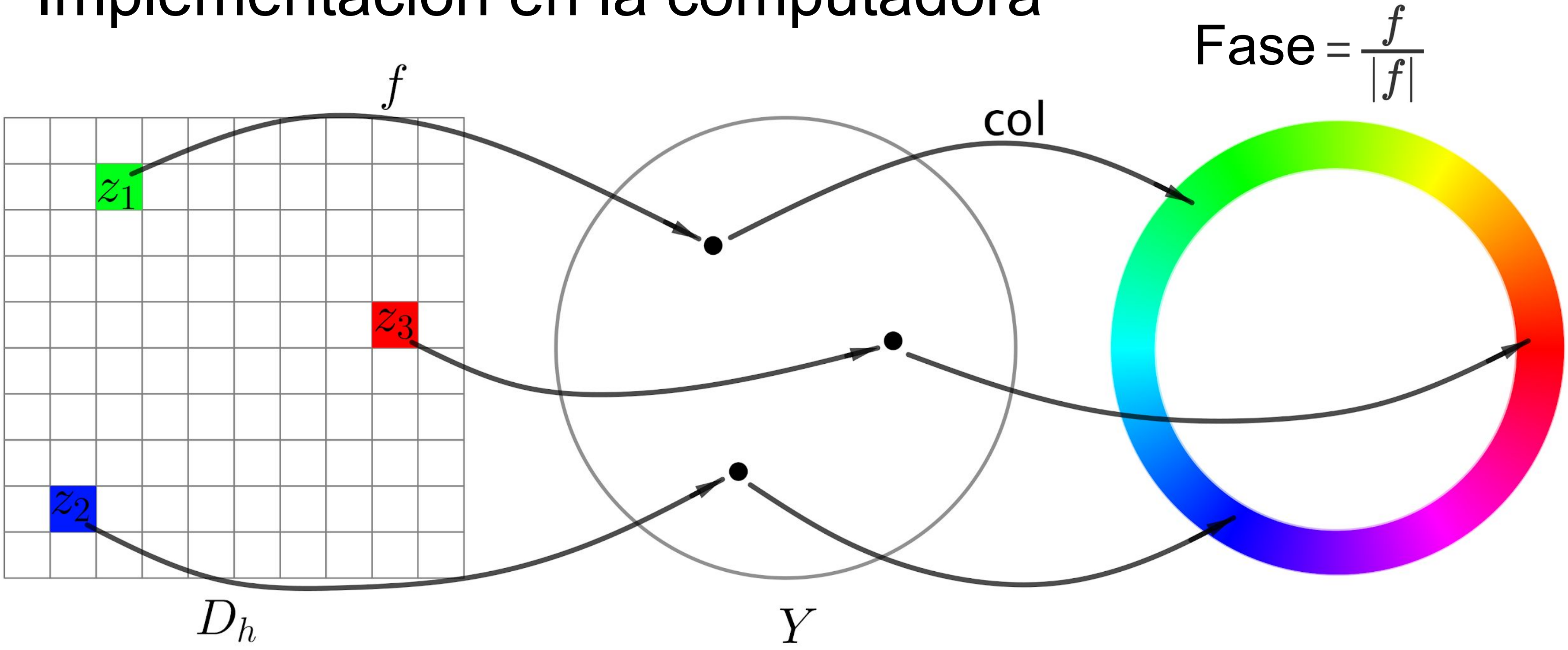
Implementación en la computadora



Implementación en la computadora



Implementación en la computadora



Implementación en la computadora

- Mathematica
- Python
- MATLAB
- Java
- C++
- JavaScript

- GeoGebra

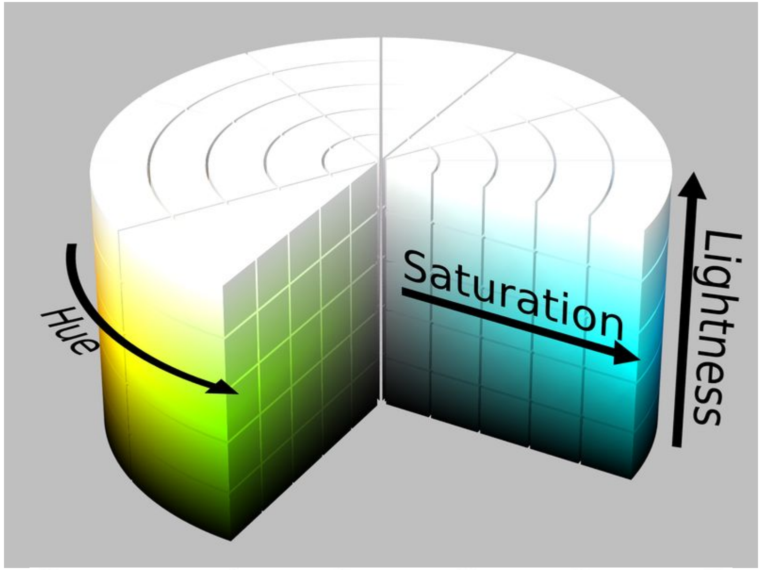
Esquema HSL

HSL: Definición de funciones en GeoGebra

H(x, y) = (π - atan2(y, -x)) / (2π)

S(x, y) = 1

L(x, y) = 0.5



$$\underbrace{[0, 2\pi]}_{H} \times \underbrace{[0, 1]}_{S} \times \underbrace{[0, 1]}_{L}$$

HSL: Definición de funciones en GeoGebra

H(x, y) = (π - atan2(y, -x)) / (2π)

S(x, y) = 1

L(x, y) = 0.5

Fase $= \frac{f}{|f|}$ = Ángulo

H(x,y) = atan2(y, x)

$$\mathbb{C} \setminus \{0\} \to (-\pi, \pi]$$

GGB - Hue: (0, 1]

$$\mathbb{C} \setminus \{0\} \to (0, 1]$$

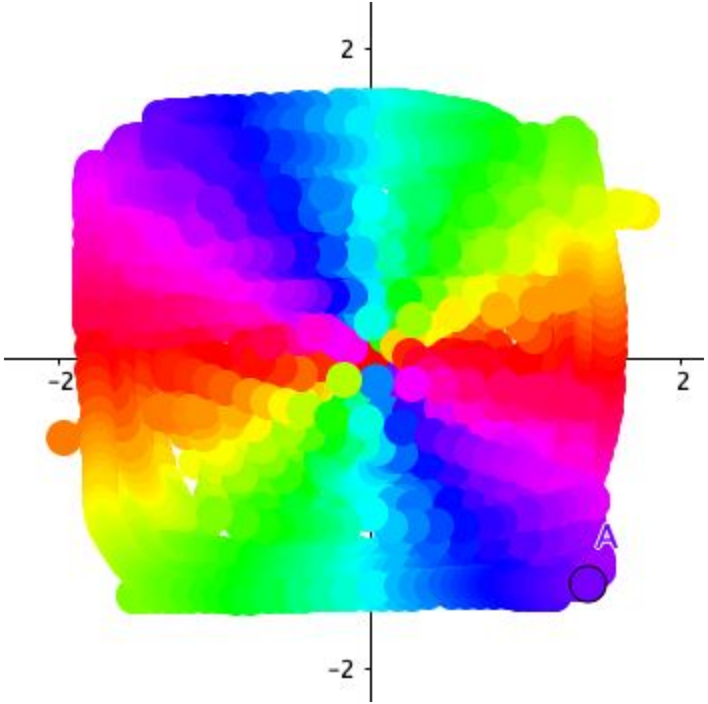
HSL: Retrato fase simple

$$f(z) = z^2$$

H(x, y) = (π - atan2(y, -x)) / (2π)
S(x, y) = 1
L(x, y) = 0.5

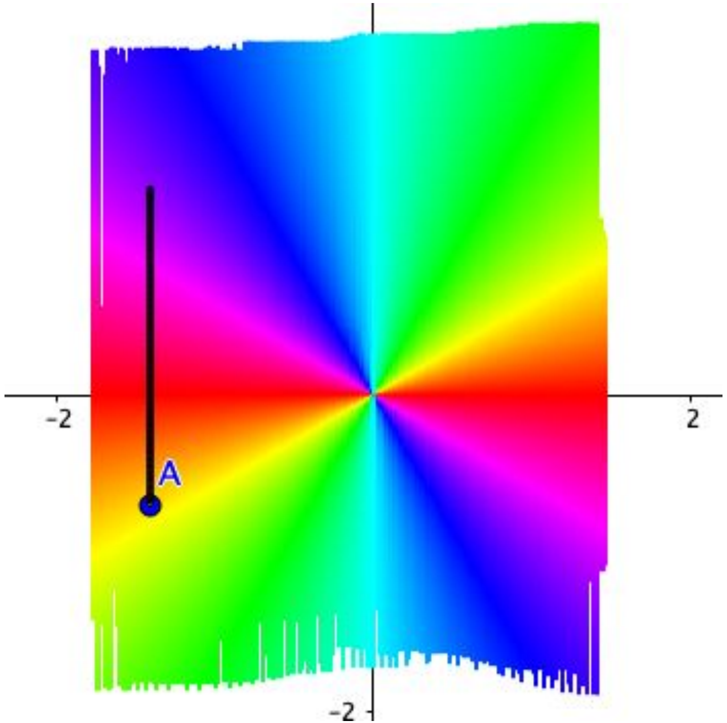
A = 1 + ί
Z = A^2

ColorDinámico(A, H(x(Z), y(Z)), S(x(Z), y(Z)), L(x(Z), y(Z)))



HSL: Escáner con hoja de cálculo $f(z) = z^2$

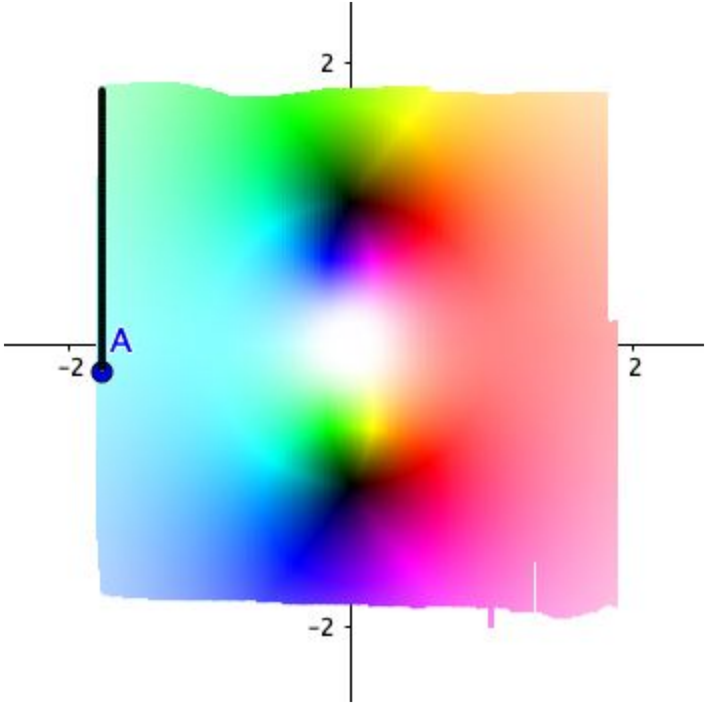
```
H(x, y) = (π -atan2(y, -x)) / (2π)
S(x, y) = 1
L(x, y) = 0.5
A = 1 + ί

A1 = 1
A2 = A1 + 1
B1 = x(A) + ί * (y(A) + A1/100)
C1 = B1^2
D1 = H(x(C1), y(C1))
E1 = S(x(C1), y(C1))
F1 = L(x(C1), y(C1))

ColorDinámico(B1, D1, E1, F1 )
```



HSL: Escáner con Módulo

$$f(z) = z + 1/z$$

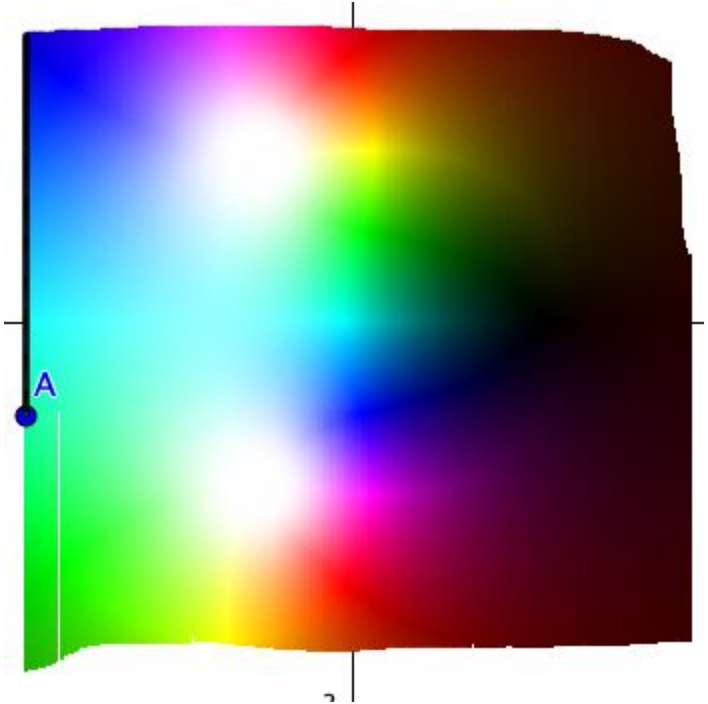
```
H(x, y) = (π -atan2(y, -x)) / (2π)
S(x, y) = 1
L(x, y) = 2 / (1 + exp(-sqrt(x^2 + y^2))) − 1
A = 1 + ί

A1 = 1
A2 = A1 + 1
B1 = x(A) + ί * (y(A) + A1/100)
C1 = B1 + 1/B1
D1 = H(x(C1), y(C1))
E1 = S(x(C1), y(C1))
F1 = L(x(C1), y(C1))

ColorDinámico(B1, D1, E1, F1 )
```



HSL: Escáner con Módulo

$$f(z) = \frac{z-1}{z^2+z+1}$$

```
H(x, y) = (π -atan2(y, -x)) / (2π)
S(x, y) = 1
L(x, y) = 2 / (1 + exp(-sqrt(x^2 + y^2))) – 1
A = 1 + ί

A1 = 1
A2 = A1 + 1
B1 = x(A) + ί * (y(A) + A1/100)
C1 = (B1 - 1)/(B1^2 + B1 +1)
D1 = H(x(C1), y(C1))
E1 = S(x(C1), y(C1))
F1 = L(x(C1), y(C1))

ColorDinámico(B1, D1, E1, F1 )
```



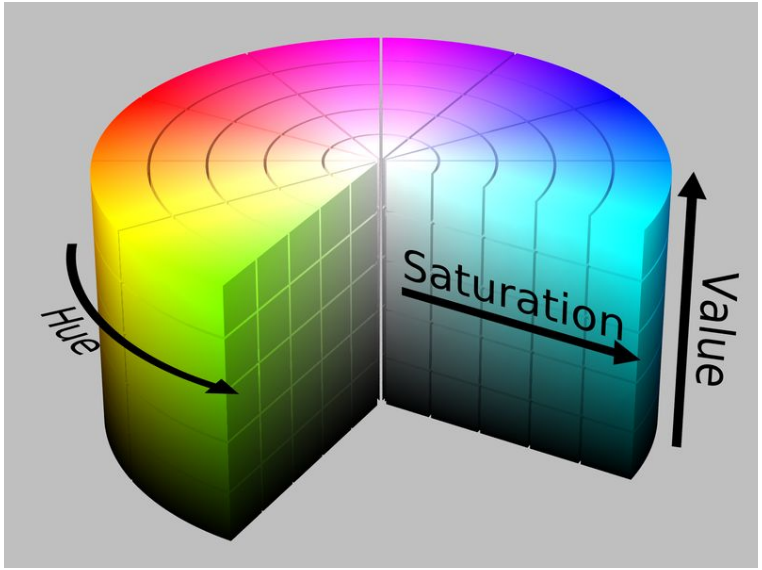
Esquema HSV

HSL: Definición de funciones en GeoGebra

H(x, y) = (π - atan2(y, -x)) / (2π)

S(x, y) = 1

V(x, y) = 1

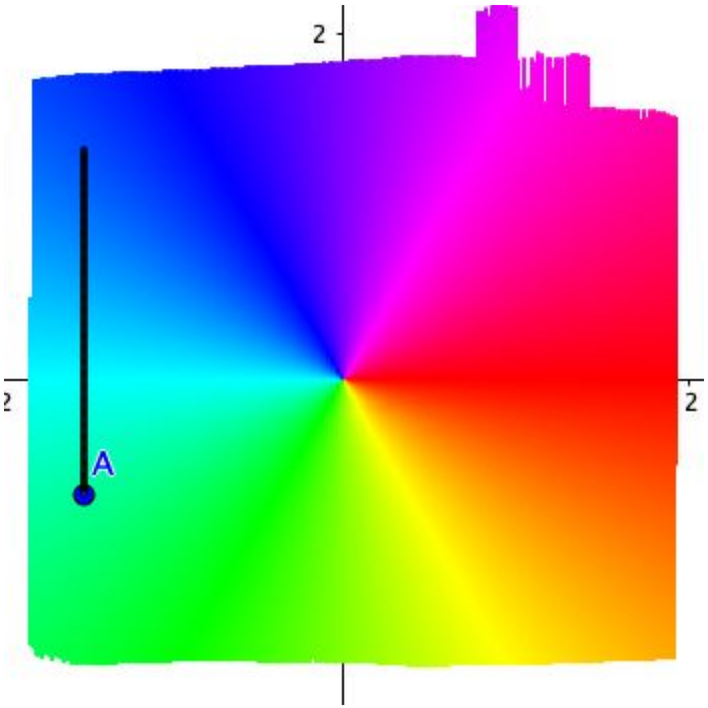


$$\underbrace{[0, 2\pi]}_{H} \times \underbrace{[0, 1]}_{S} \times \underbrace{[0, 1]}_{L}$$

HSV: Retrato fase simple

$$f(z) = 1/z$$

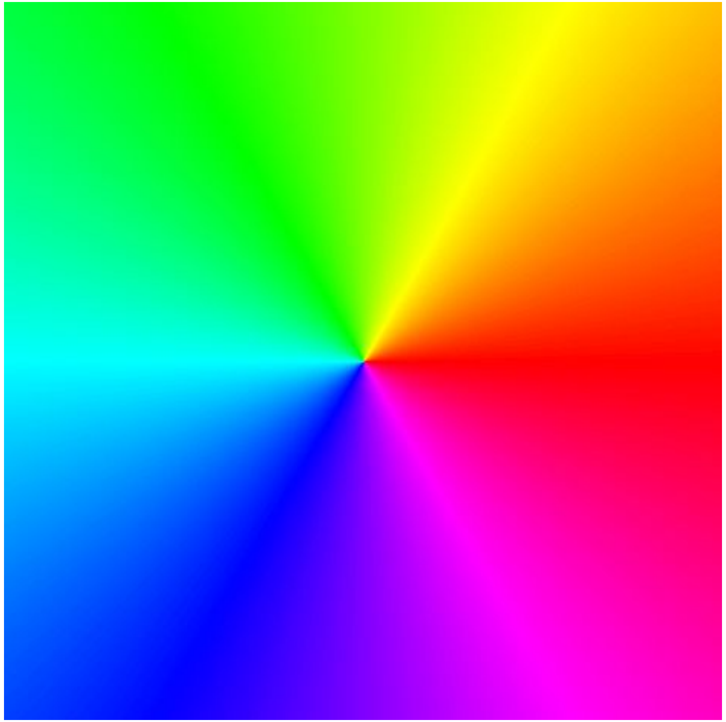
```
H(x, y) = (π -atan2(y, -x)) / (2π)
S(x, y) = 1
V(x, y) = 1
A = 1 + ί

A1 = 1
A2 = A1 + 1
B1 = x(A) + ί * (y(A) + A1/100)
C1 = 1/B1
D1 = H(x(C1), y(C1))
E1 = S(x(C1), y(C1))
F1 = V(x(C1), y(C1))

ColorDinámico(B1, D1, E1, F1 )
```



Retrato fase simple

$$[-2, 2] \times [-2, 2]$$

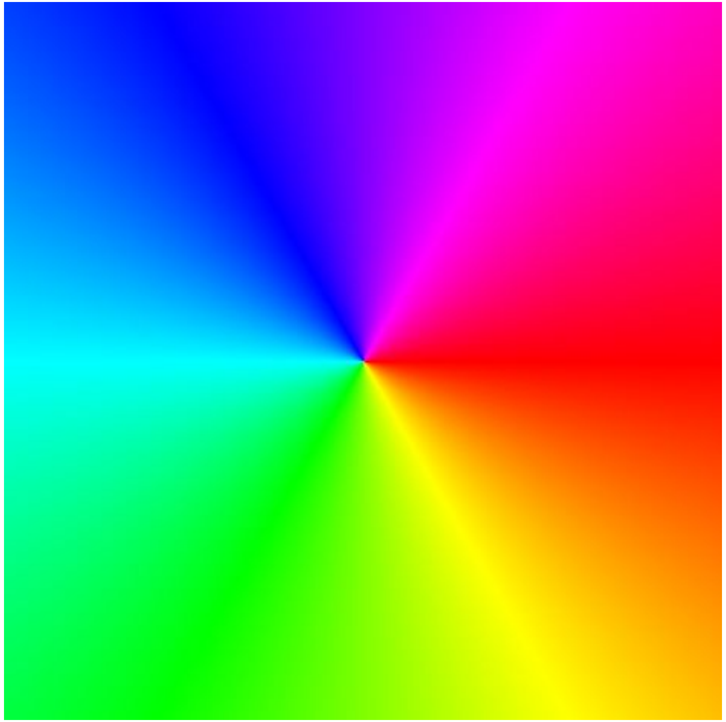
$$f(z) = z$$



Retrato fase simple

$$[-2,2] \times [-2,2]$$

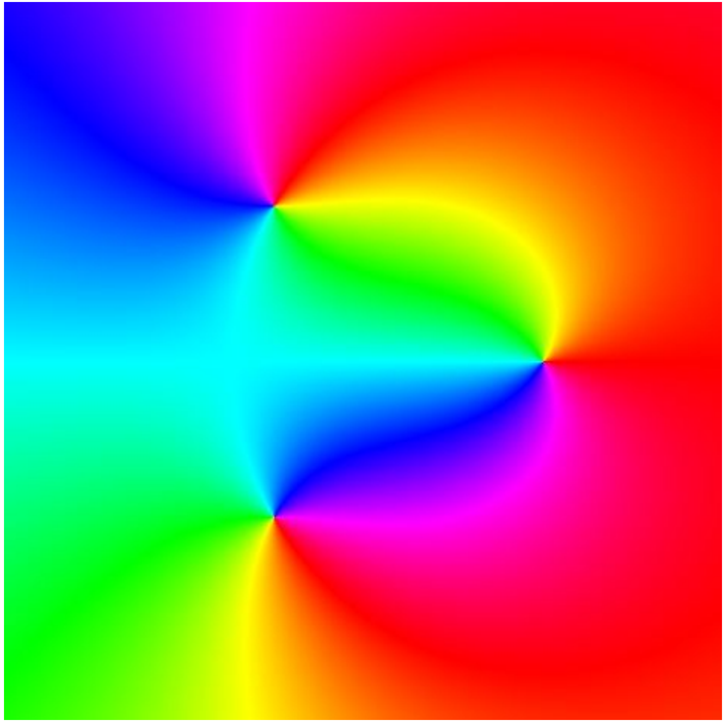
$$f(z) = 1/z$$



Retrato fase simple

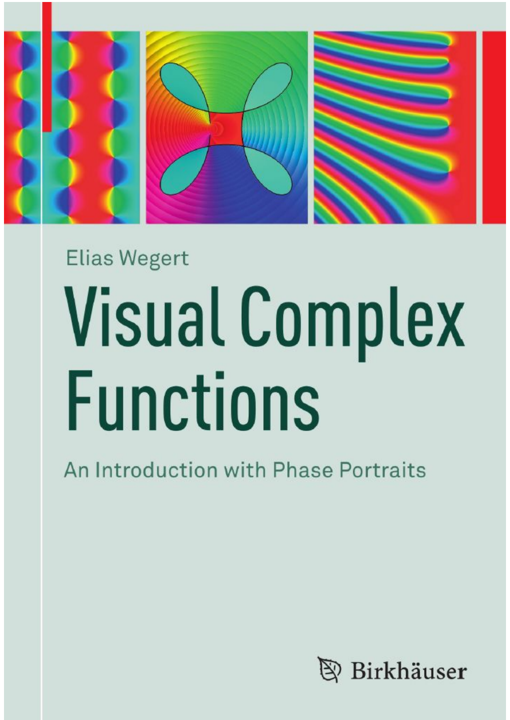
$$f(z) = \frac{z-1}{z^2+z+1}$$

$$[-2,2] \times [-2,2]$$

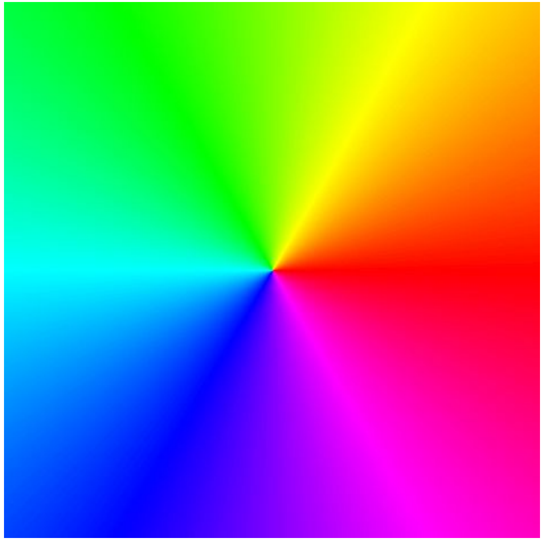


Retrato fase mejorado

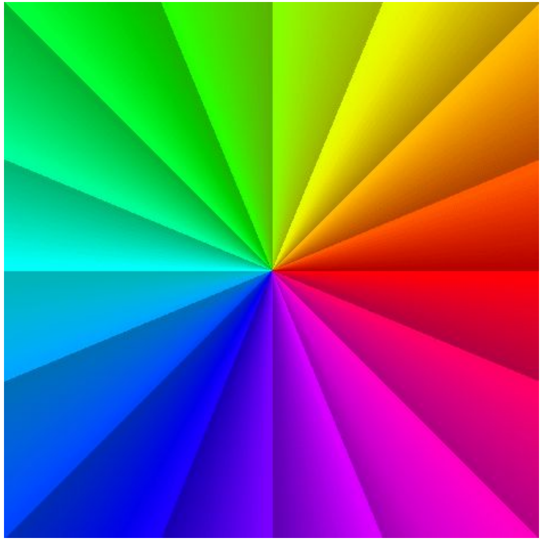
Elias Wegert's work from 2012:



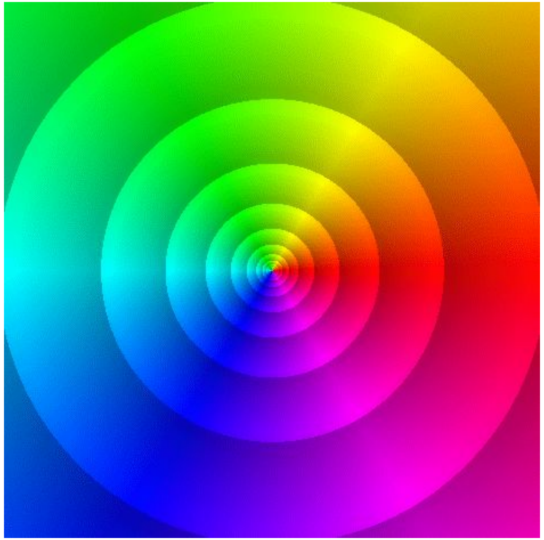
Retrato fase mejorado



$$f(z) = z$$

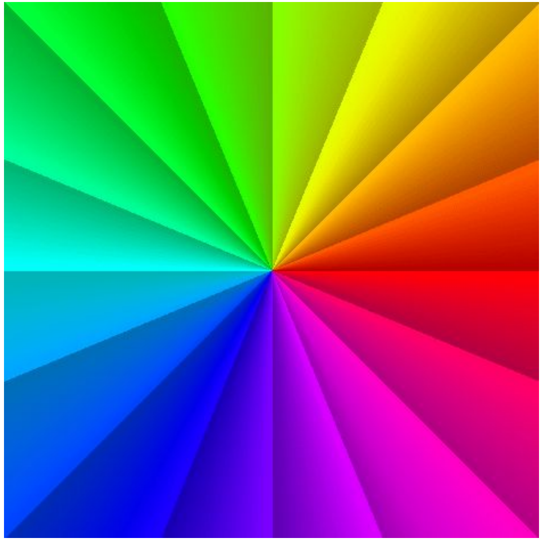


Fase

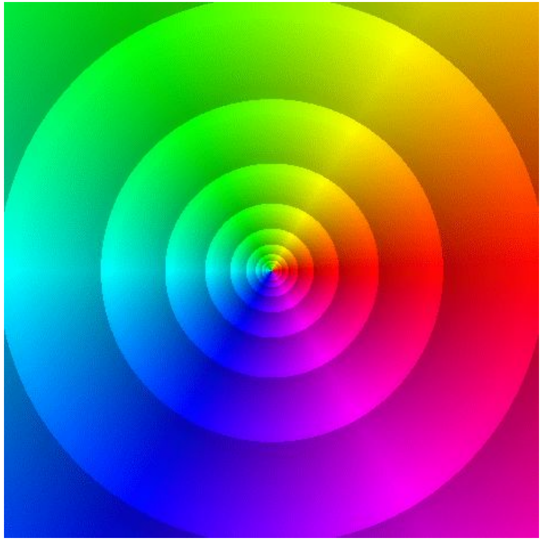


Módulo

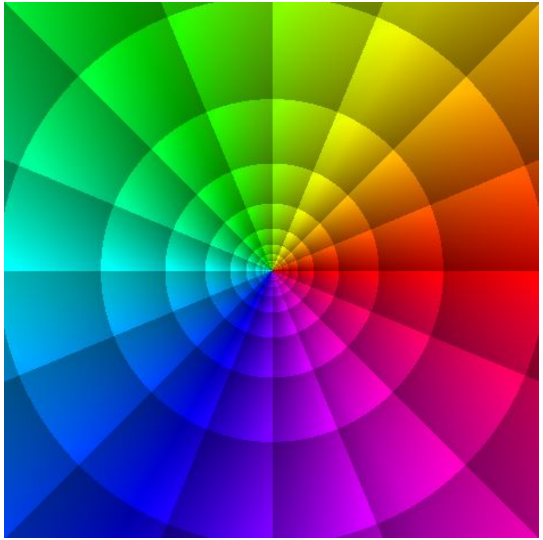
Retrato fase mejorado



Fase

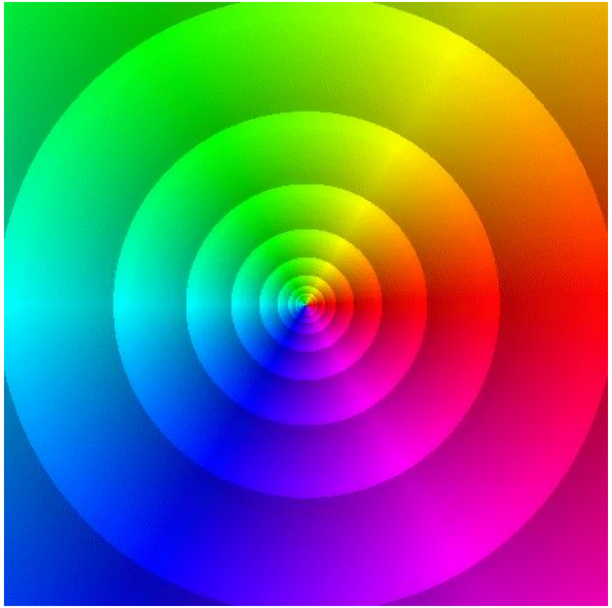


Módulo

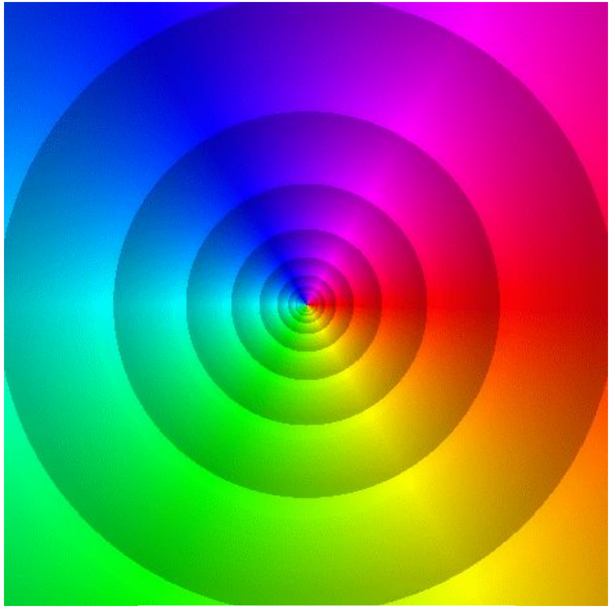


Combinado

Retrato fase mejorado: Módulo

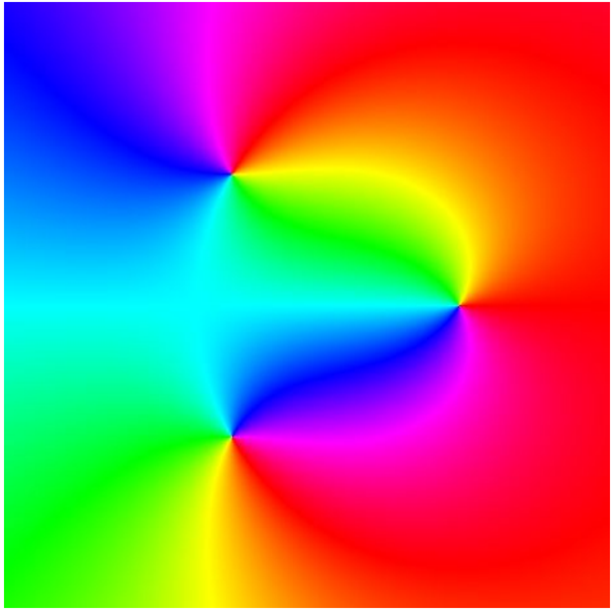


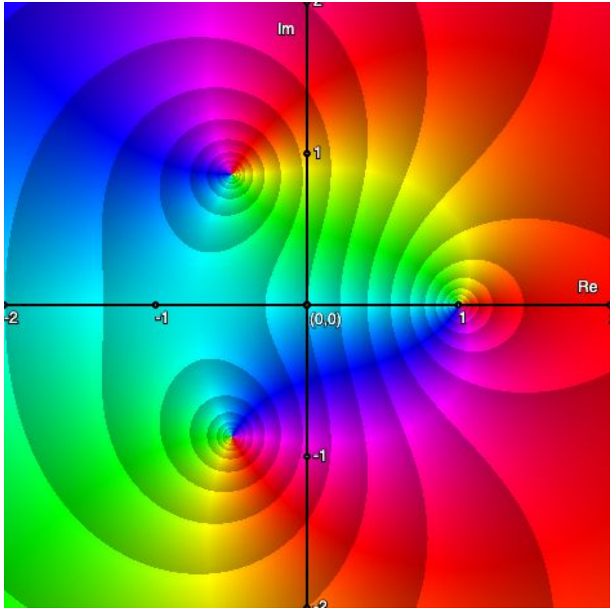
$f(z) = z$



$f(z) = 1/z$

Retrato fase mejorado de $f(z) = \dfrac{z-1}{z^2+z+1}$





$$[-2,2] \times [-2,2]$$

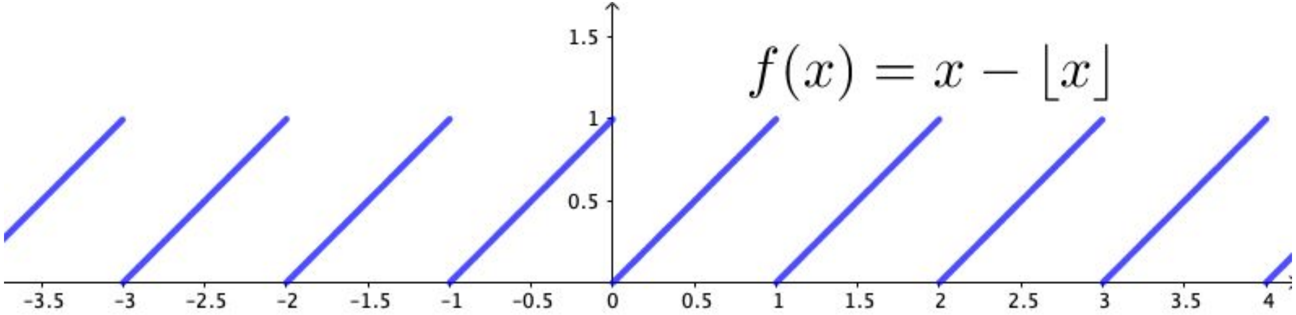
¿Cómo introducir las curvas de nivel?

Fase = $(\pi - \text{atan2}(y, -x)) / (2\pi)$

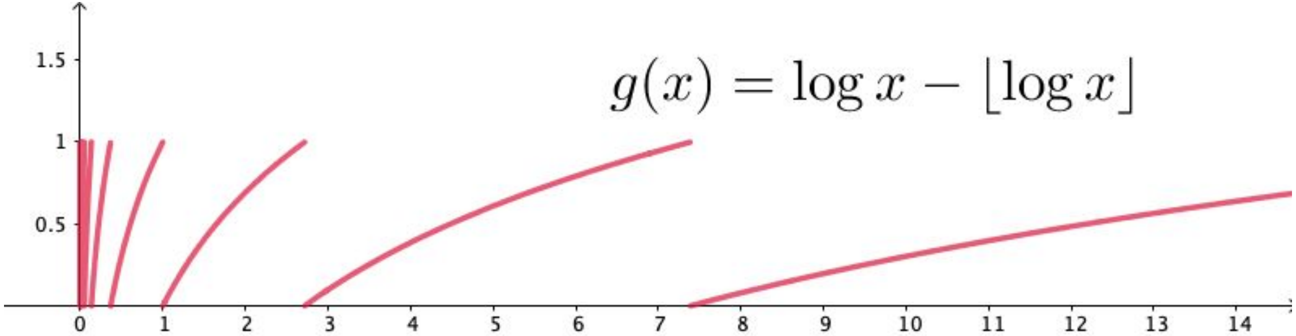
Módulo = sqrt(x^2+y^2)

¿Cómo introducir las curvas de nivel?

Fase:

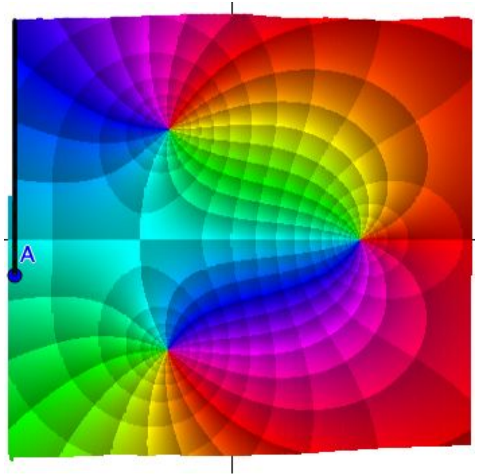


Módulo:



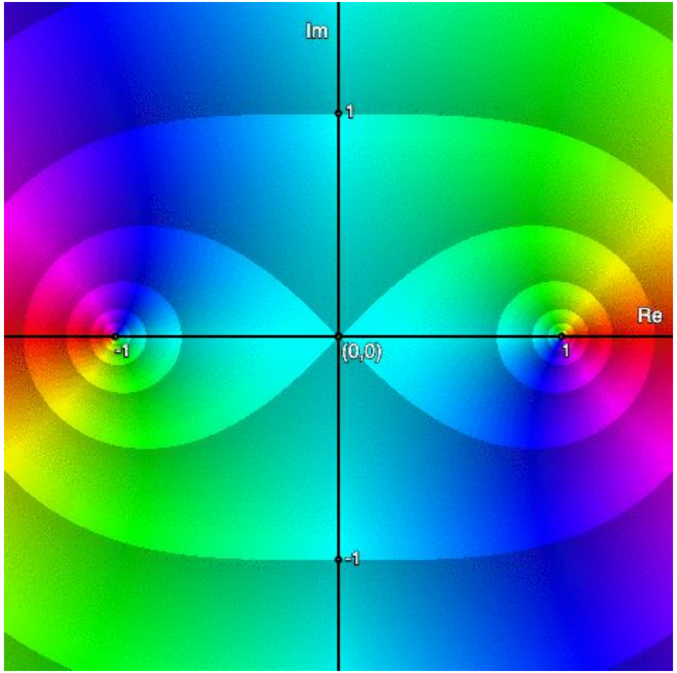
Esquema HSV - Retrato de fase mejorado

```
H(x, y) = (π -atan2(y,-x)) / (2π)
S(x, y) = 1 / 3 (18(π - atan2(y,-x)) / (2π) - floor(18(π - atan2(y,-x)) / (2π))) + 0.7
V(x, y) = log(1.5, sqrt(x^2 + y^2) 5) / 5 - floor(log(1.5, sqrt(x^2 + y^2) 5)) / 5 + 0.8
A = 1 + ί

A1 = 1
A2 = A1 + 1
B1 = x(A) + ί * (y(A) + A1/100)
C1 = (B1-1)/(B1^2+B1+1)
D1 = H(x(C1), y(C1))
E1 = S(x(C1), y(C1))
F1 = V(x(C1), y(C1))

ColorDinámico(B1, D1, 1, E1 F1 )
```



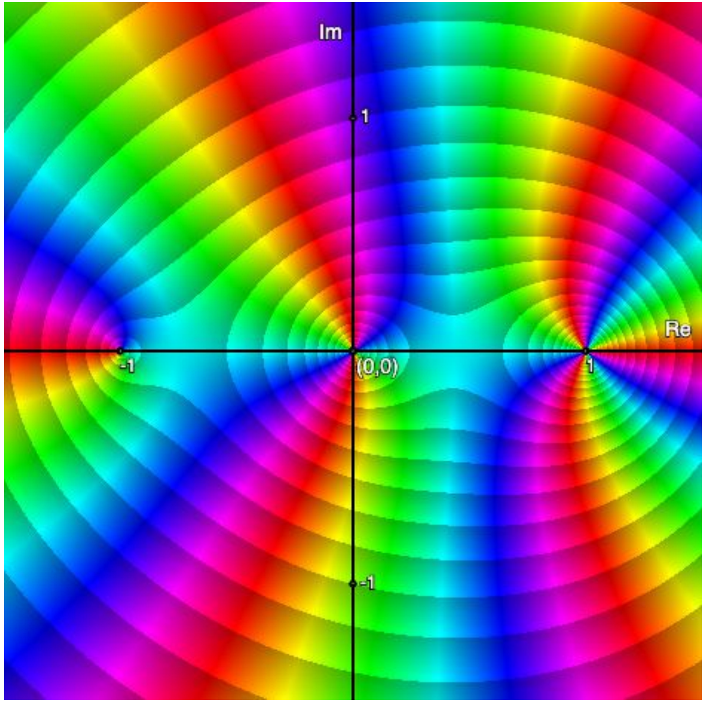
Más ejemplos para explorar

Raíces de la unidad: $z^n - 1, \quad n = 2, 3, 4, \ldots$



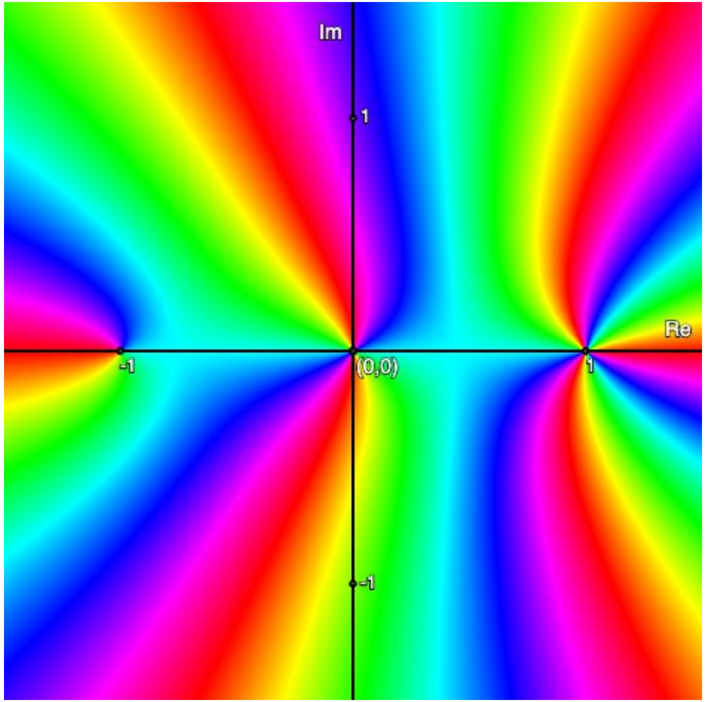
$[-1.5, 1.5] \times [-1.5, 1.5]$

Multiplicidad de ceros: $f(z) = (z+1)z^2(z-1)^3$



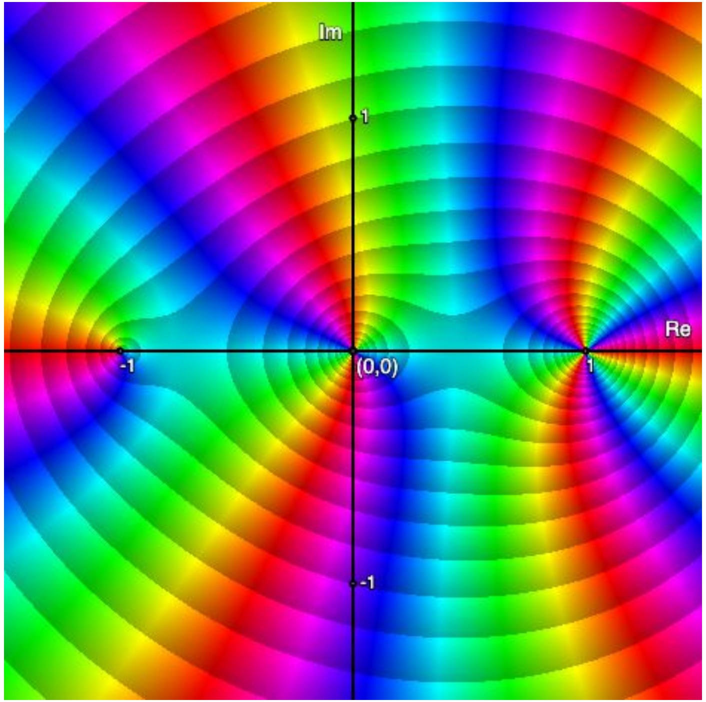
$[-1.5, 1.5] \times [-1.5, 1.5]$

Multiplicidad de ceros: $f(z) = (z+1)z^2(z-1)^3$



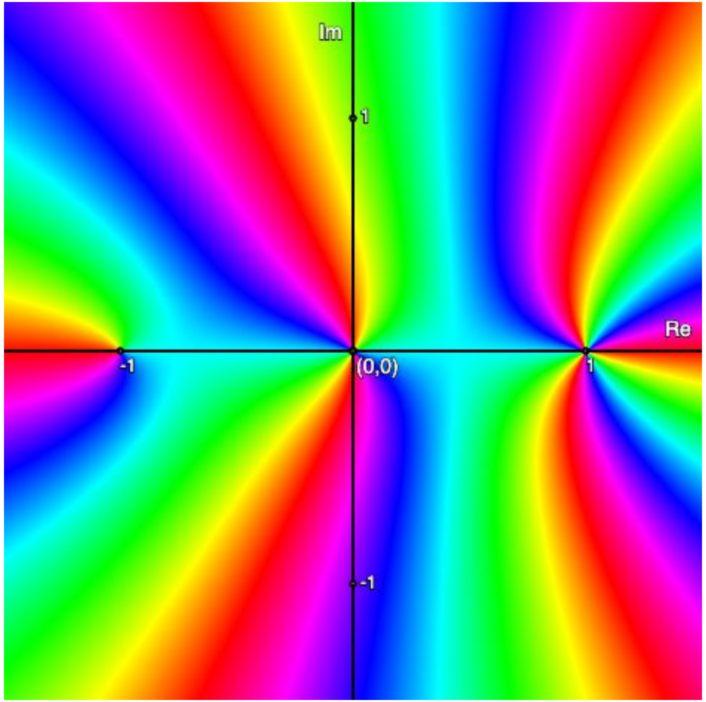
$[-1.5, 1.5] \times [-1.5, 1.5]$

Orden de poles: $f(z) = \frac{1}{(z+1)z^2(z-1)^3}$



$[-1.5, 1.5] \times [-1.5, 1.5]$

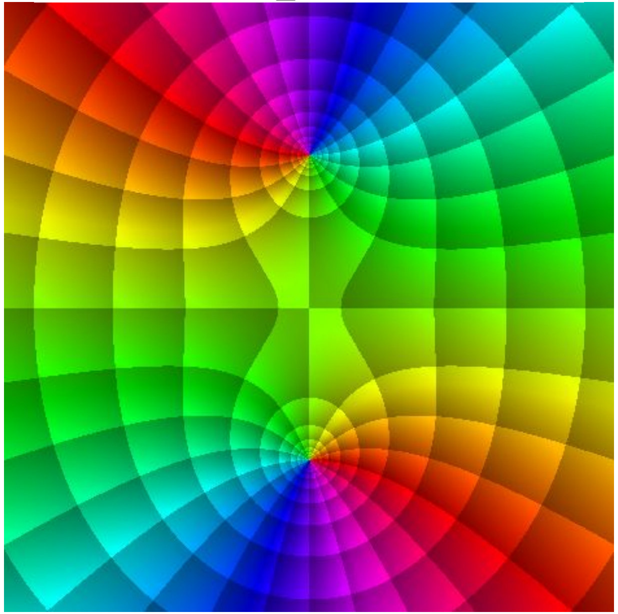
Orden de poles: $f(z) = \frac{1}{(z+1)z^2(z-1)^3}$



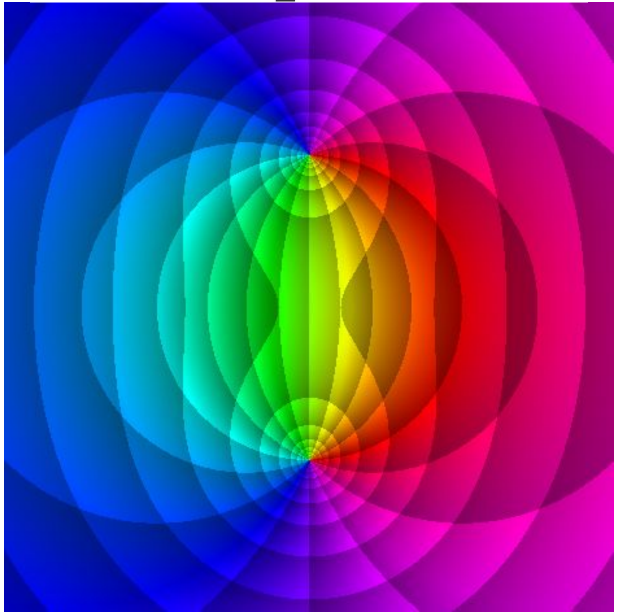
$[-1.5, 1.5] \times [-1.5, 1.5]$

Funciones analíticas y no-analíticas

$$f(z) = \frac{3}{2} z(1 - iz)$$



$$g(z) = \frac{3}{2} z(1 - i\bar{z})$$



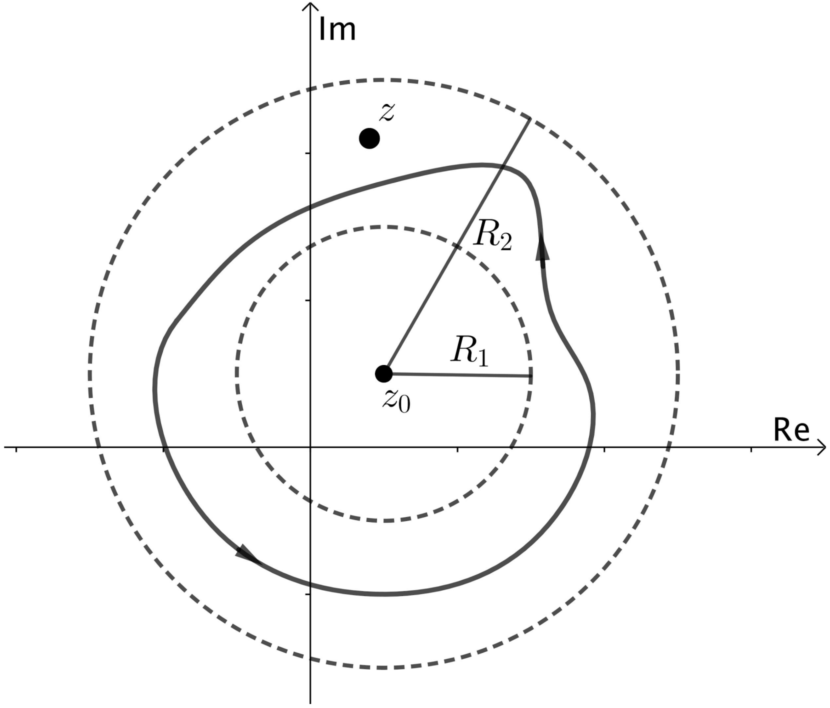
Series de Laurent

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$(R_1 < |z - z_0| < R_2)$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots)$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{-n+1}} \quad (n = 1, 2, \ldots)$$

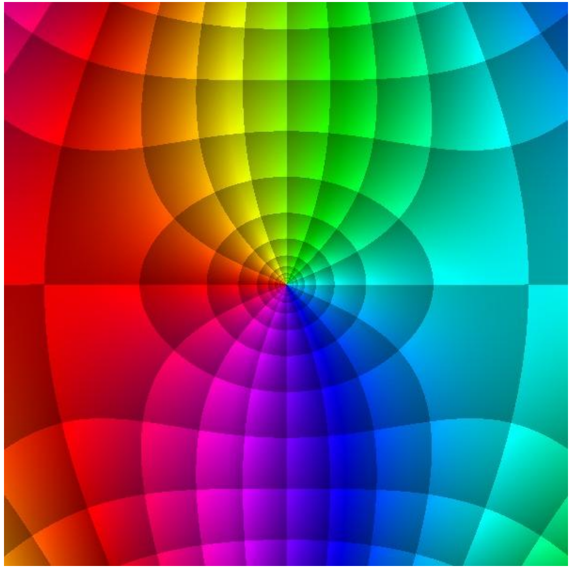


Series de Laurent

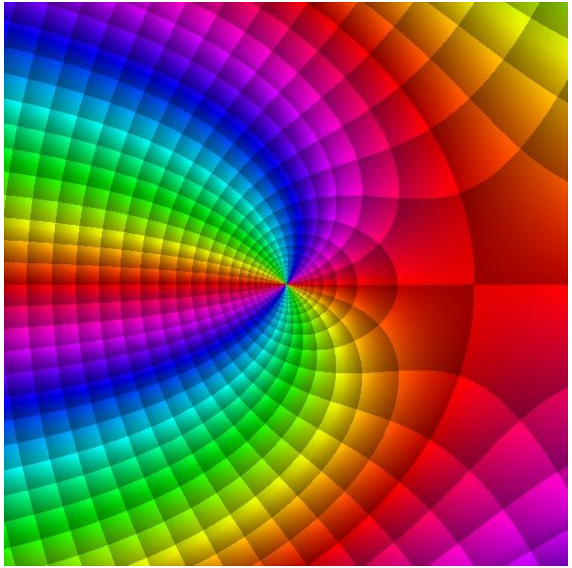
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \boxed{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}$$

Clasificación de singularidades

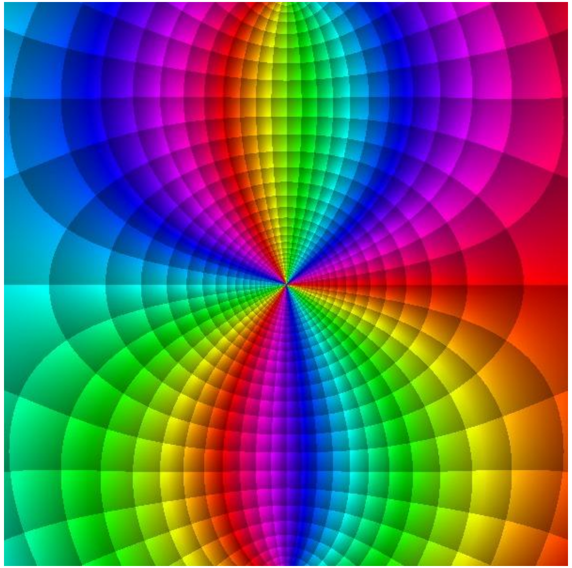
Polos de orden m $\exists\ m \geq 1,\ \ b_m \neq 0$ and $b_k = 0$ for $k > m$



$$\frac{1 - \cosh z}{z^3}$$

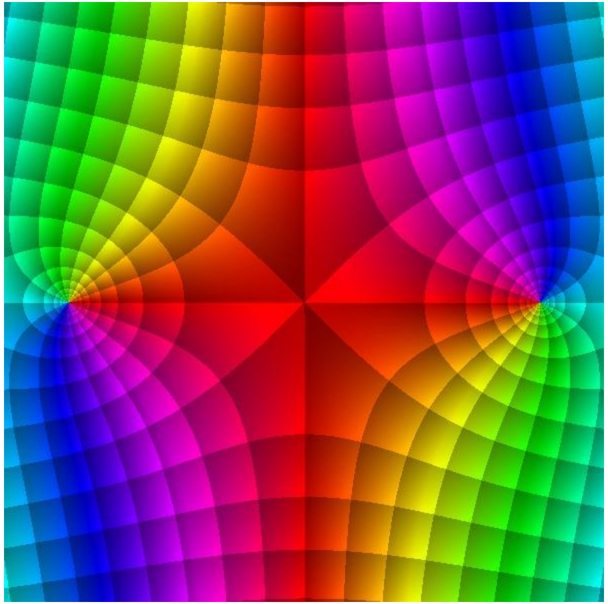


$$\frac{\exp(2z)}{(z-1)^2}$$

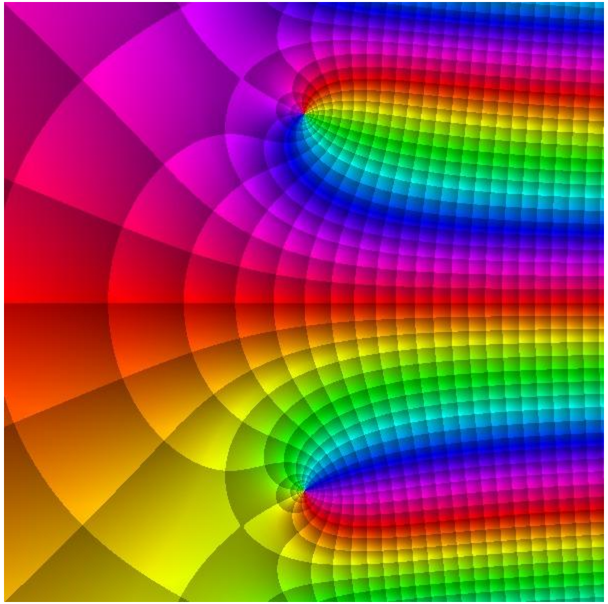


$$\frac{\sinh z}{z^4}$$

Singularities removibles: $b_n = 0, \forall n$

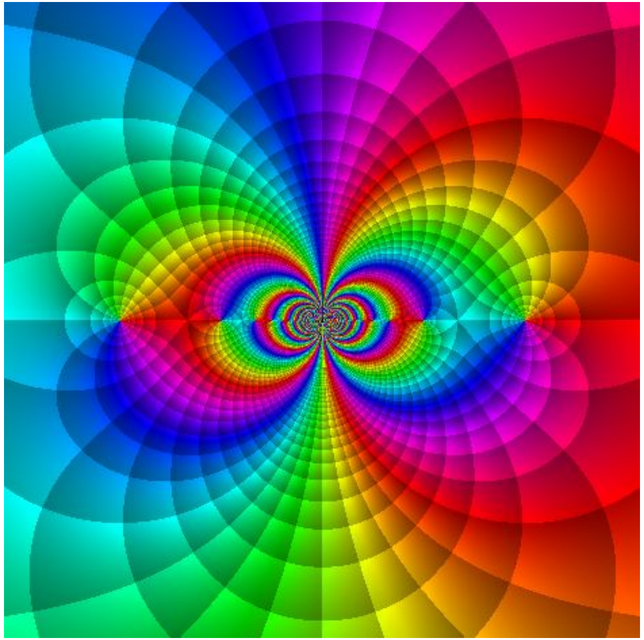


$$\frac{\sin z}{z}$$

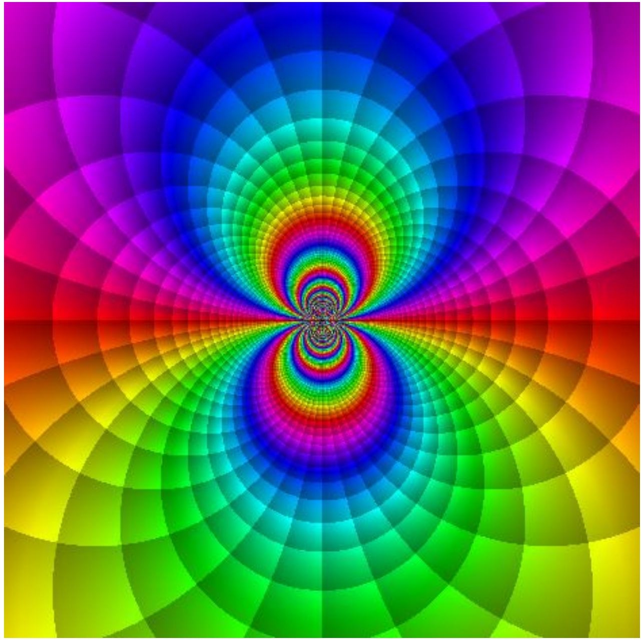


$$\frac{z}{e^z - 1}$$

Singularidades esenciales



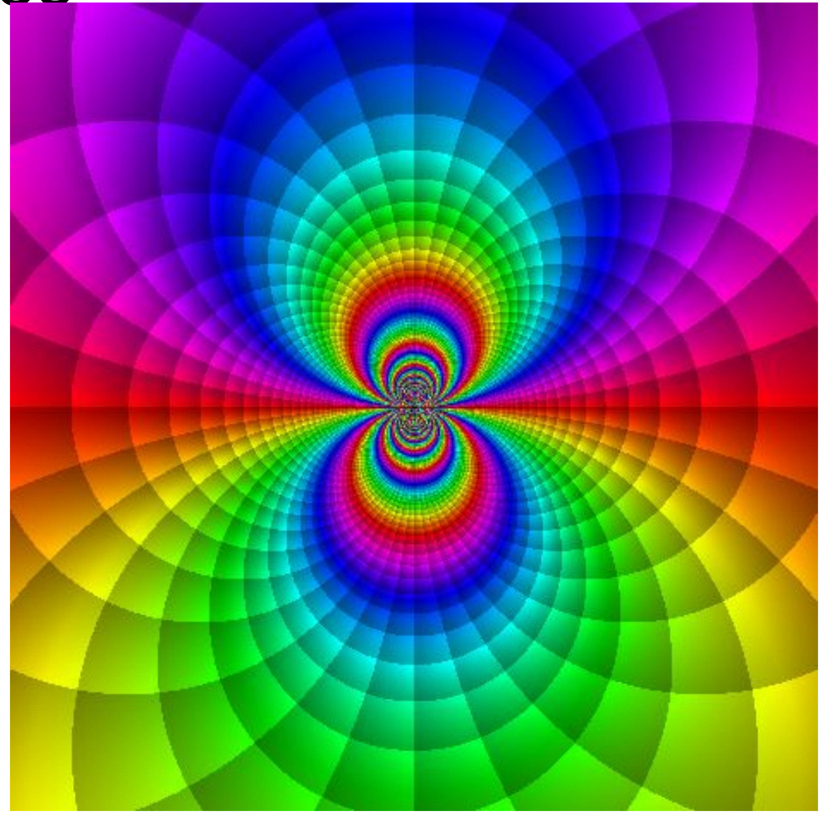
$$\sin\left(\frac{1}{z}\right)$$



$$\exp\left(\frac{1}{z}\right)$$

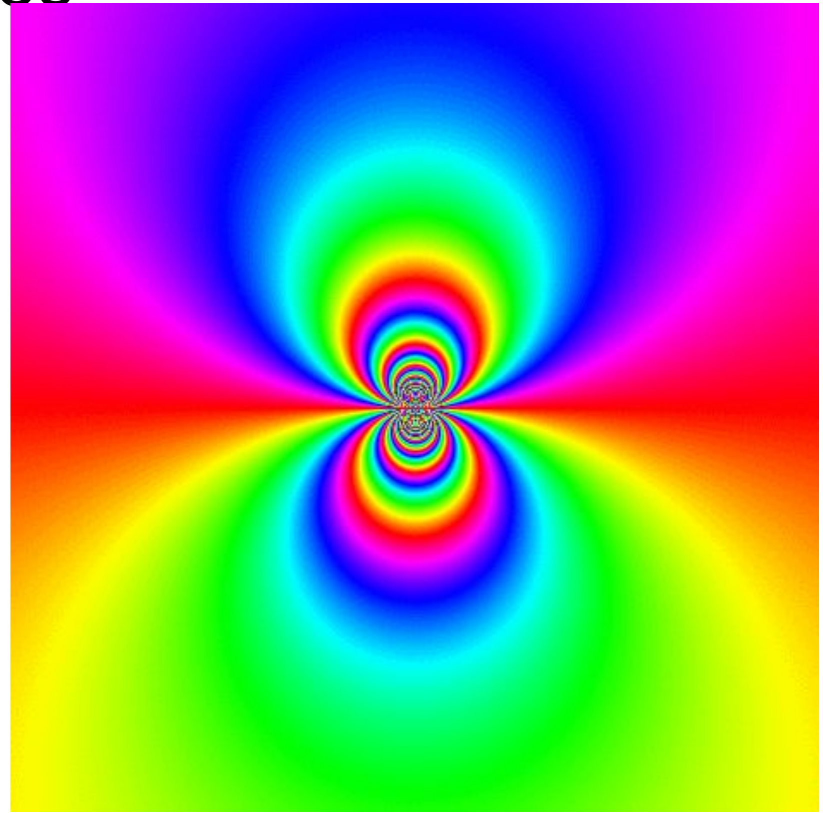
Singularidades esenciales

$$\exp\left(\frac{1}{z}\right)$$



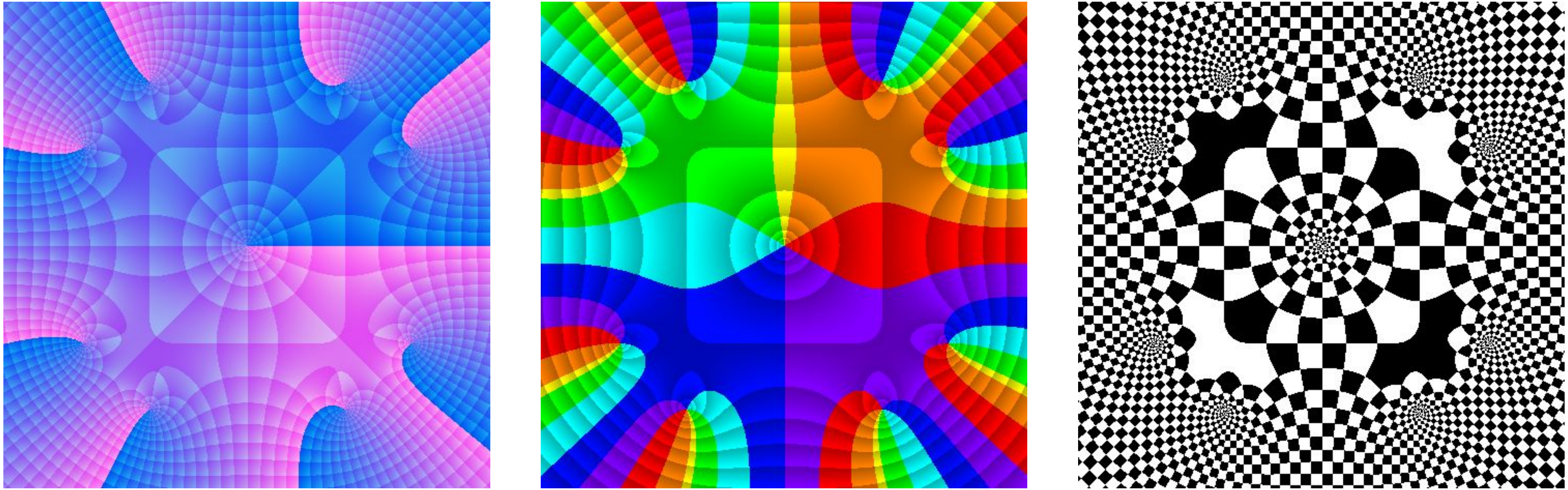
Singularidades esenciales

$$\exp\left(\frac{1}{z}\right)$$



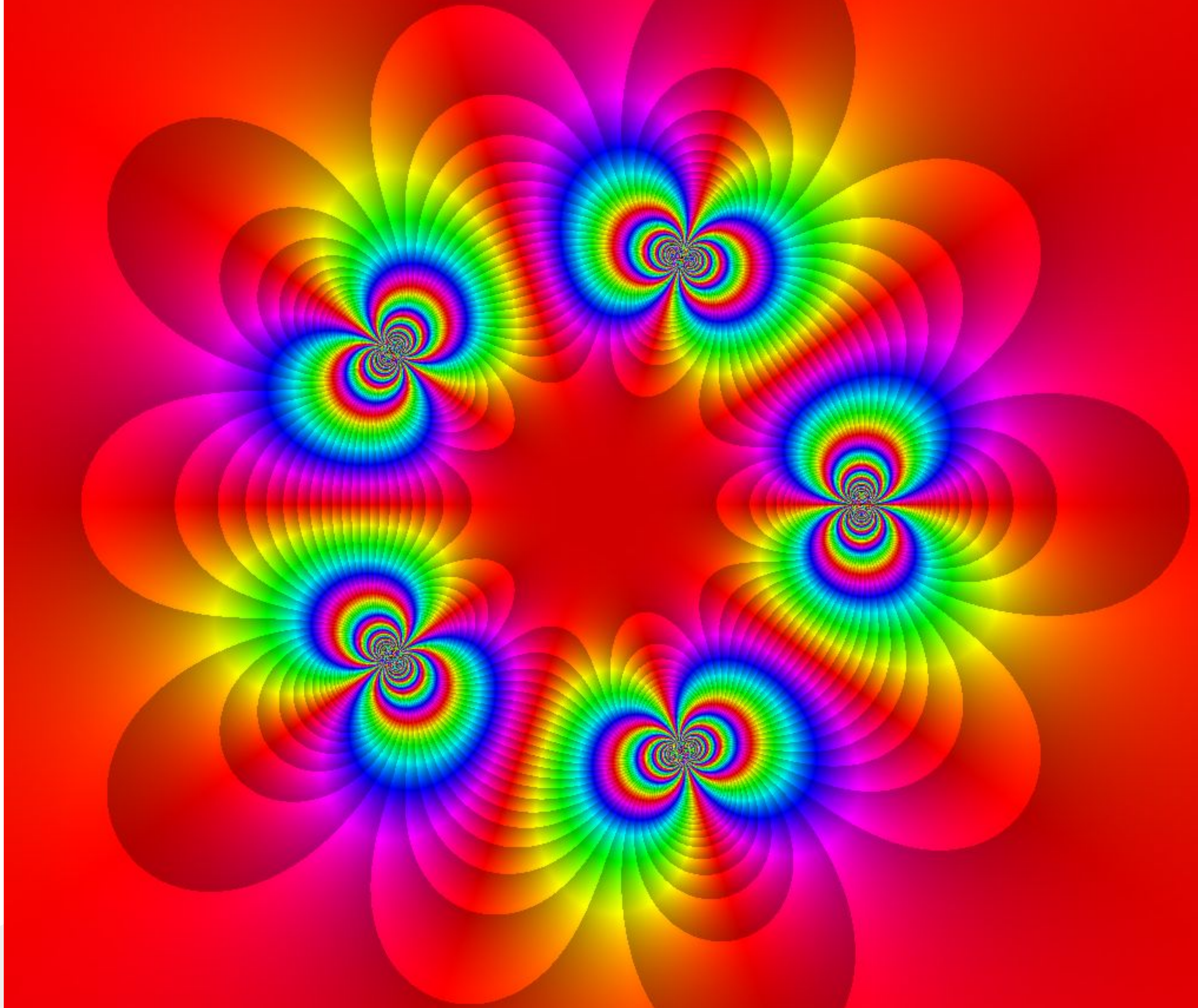
Otros esquemas de color...

$$f(z) = 0.926(z + 0.073857z^5 + 0.0045458z^9)$$



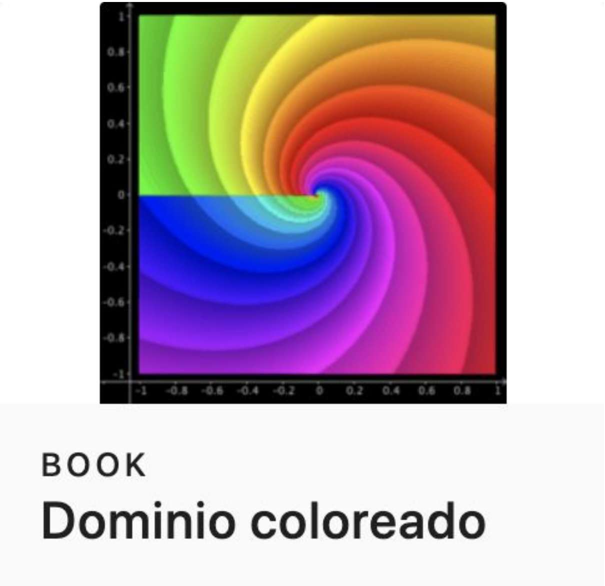
¡Gracias!

$$f(z) = \prod_{k=1}^{5} \exp\left(\frac{z + \omega^k}{z - \omega^k}\right)$$



Recursos en línea

https://www.geogebra.org/m/DdncMp6t



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