IDENTITAS

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Mata Kuliah : Aplikasi Komputer

Kalkulus dengan EMT

Materi Kalkulus mencakup di antaranya:

- Fungsi (fungsi aljabar, trigonometri, eksponensial, logaritma, komposisi fungsi)
- Limit Fungsi,
- Turunan Fungsi,
- Integral Tak Tentu,
- Integral Tentu dan Aplikasinya,
- Barisan dan Deret (kekonvergenan barisan dan deret).

EMT (bersama Maxima) dapat digunakan untuk melakukan semua perhitungan di dalam kalkulus, baik secara numerik maupun analitik (eksak).

Mendefinisikan Fungsi

Terdapat beberapa cara mendefinisikan fungsi pada EMT, yakni:

- Menggunakan format nama_fungsi := rumus fungsi (untuk fungsi numerik),
- Menggunakan format nama_fungsi &= rumus fungsi (untuk fungsi simbolik, namun dapat dihitung secara numerik),
- Menggunakan format nama_fungsi &&= rumus fungsi (untuk fungsi simbolik murni, tidak dapat dihitung langsung),
- Fungsi sebagai program EMT.

Setiap format harus diawali dengan perintah function (bukan sebagai ekspresi).

Berikut adalah adalah beberapa contoh cara mendefinisikan fungsi:

$$f(x) = 2x^2 + e^{\sin(x)}.$$

```
>function f(x) := 2*x^2+exp(sin(x)) // fungsi numerik
>f(0), f(1), f(pi)
```

```
  1
  4.31977682472
  20.7392088022
```

```
>f(a) // tidak dapat dihitung nilainya
```

```
  Variable or function a not found.
  Error in:
  f(a) // tidak dapat dihitung nilainya ...
     ^
```

Silakan Anda plot kurva fungsi di atas!
Berikutnya kita definisikan fungsi:

$$g(x) = \frac{\sqrt{x^2 - 3x}}{x + 1}.$$

```
>function g(x) := sqrt(x^2-3*x)/(x+1)
>g(3)
```

```
  0
```

```
>g(0)
```

```
  0
```

```
>g(1) // kompleks, tidak dapat dihitung oleh fungsi numerik
```

```
  Floating point error!
  Error in sqrt
  Try "trace errors" to inspect local variables after errors.
  g:
      useglobal; return sqrt(x^2-3*x)/(x+1)
  Error in:
  g(1) // kompleks, tidak dapat dihitung oleh fungsi numerik ...
      ^
```

Silakan Anda plot kurva fungsi di atas!

```
>f(g(5)) // komposisi fungsi
```

```
  2.20920171961
```

```
>g(f(5))
```

```
  0.950898070639
```

```
>function h(x) := f(g(x)) // definisi komposisi fungsi
>h(5) // sama dengan f(g(5))
```

```
  2.20920171961
```

Silakan Anda plot kurva fungsi komposisi fungsi f dan g:

$$h(x) = f(g(x))$$

dan

$$u(x) = g(f(x))$$

bersama-sama kurva fungsi f dan g dalam satu bidang koordinat.

```
>f(0:10) // nilai-nilai f(0), f(1), f(2), ..., f(10)
```

```
  [1,  4.31978,  10.4826,  19.1516,  32.4692,  50.3833,  72.7562,
  99.929,  130.69,  163.51,  200.58]
```

```
>fmap(0:10) // sama dengan f(0:10), berlaku untuk semua fungsi
```

```
  [1,  4.31978,  10.4826,  19.1516,  32.4692,  50.3833,  72.7562,
  99.929,  130.69,  163.51,  200.58]
```

```
>gmap(200:210)
```

```
  [0.987534,  0.987596,  0.987657,  0.987718,  0.987778,  0.987837,
  0.987896,  0.987954,  0.988012,  0.988069,  0.988126]
```

Misalkan kita akan mendefinisikan fungsi

$$f(x) = \begin{cases} x^3 & x > 0 \\ x^2 & x \leq 0. \end{cases}$$

Fungsi tersebut tidak dapat didefinisikan sebagai fungsi numerik secara "inline" menggunakan format :=, melainkan didefinisikan sebagai program. Perhatikan, kata "map" digunakan agar fungsi dapat menerima vektor sebagai input, dan hasilnya berupa vektor. Jika tanpa kata "map" fungsinya hanya dapat menerima input satu nilai.

```
>function map f(x) ...
```

```
    if x>0 then return x^3
    else return x^2
    endif;
  endfunction
```

```
>f(1)
```

```
  1
```

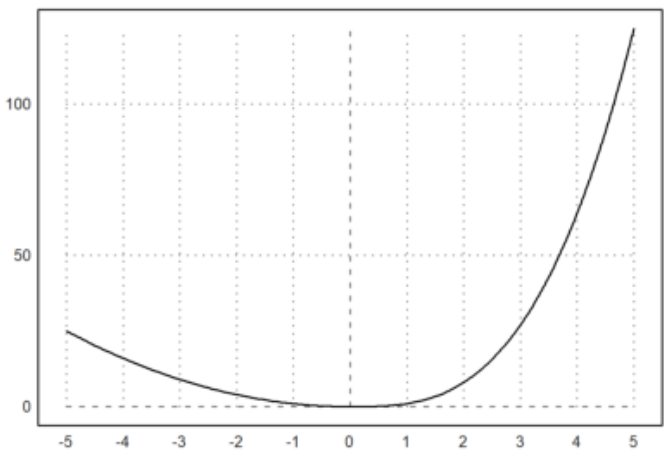
```
>f(-2)
```

```
  4
```

```
>f(-5:5)
```

```
  [25,  16,  9,  4,  1,  0,  1,  8,  27,  64,  125]
```

```
>aspect(1.5); plot2d("f(x)",-5,5):
```




```
>function f(x) &= 2*E^x // fungsi simbolik
```

$$2\,E^{x}$$

```
>$f(a) // nilai fungsi secara simbolik
```

$$2\,e^{a}$$

```
>f(E) // nilai fungsi berupa bilangan desimal
```

```
  30.308524483
```

```
>$f(E), $float(%)
```

$$2\,e^{e}$$

$$30.30852448295852$$

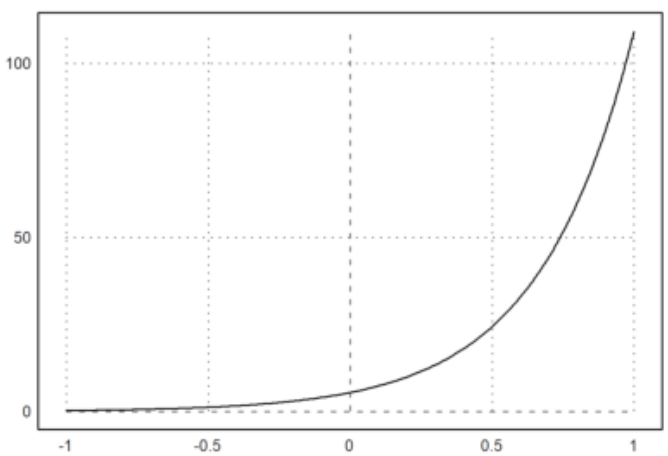
```
>function g(x) &= 3*x+1
```

$$3\ x + 1$$

```
>function h(x) &= f(g(x)) // komposisi fungsi
```

$$\begin{array}{c} 3\ x + 1 \\ 2\ E \end{array}$$

```
>plot2d("h(x)",-1,1):
```



Latihan

Bukalah buku Kalkulus. Cari dan pilih beberapa (paling sedikit 5 fungsi berbeda tipe/bentuk/jenis) fungsi dari buku tersebut, kemudian definisikan fungsi-fungsi tersebut dan komposisinya di EMT pada baris-baris perintah berikut (jika perlu tambahkan lagi). Untuk setiap fungsi, hitung beberapa nilainya, baik untuk satu nilai maupun vektor. Gambar grafik fungsi-fungsi tersebut dan komposisi-komposisi 2 fungsi.

Juga, carilah fungsi beberapa (dua) variabel. Lakukan hal sama seperti di atas.

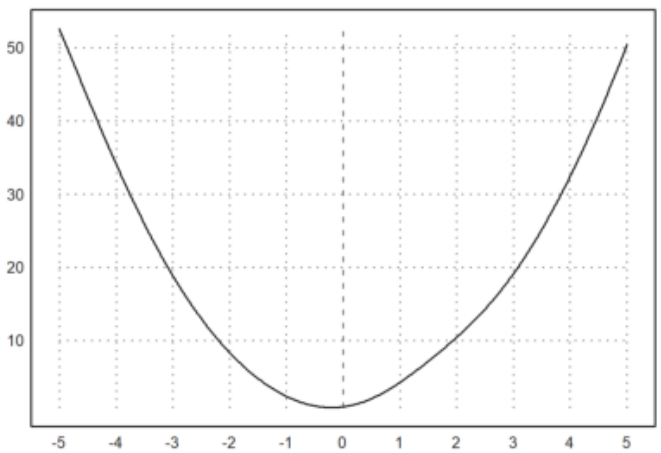
```
>function f(x) := 2*x^2 + exp(sin(x));
```

Fungsi polinomial + eksponensial (numerik)

```
>f(0); f(1); f(0:10);
```

:= membuat fungsi numerik (bisa dievaluasi langsung).
exp(sin(x)) adalah e^{sin x}.
f(0:10) memanggil f pada vektor 0..10 dan menghasilkan vektor hasil (vektor otomatis).
Contoh output (dari file): f(0) = 1, f(1) ~ 4.31977682472.

```
>aspect(1.5); plot2d("f(x)", -5, 5):
```




```
>function g(x) := sqrt(x^2 - 3*x)/(x + 1);
```

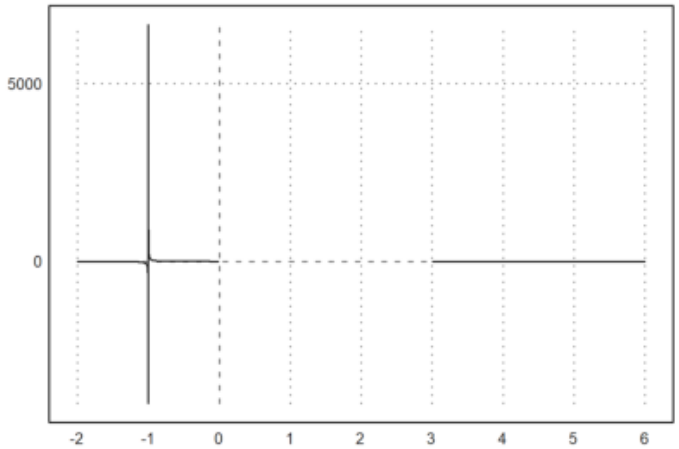
Fungsi rasional (dengan domain terbatas)

```
>g(3); g(0); g(1);
```

```
  Floating point error!
  Error in sqrt
  Try "trace errors" to inspect local variables after errors.
  g:
      useglobal; return sqrt(x^2 - 3*x)/(x + 1);
  Error in:
  g(3); g(0); g(1); ...
                  ^
```

Perhatikan domain: sqrt(x^2 - 3*x) memerlukan ekspresi = 0; pembagi x+1 tidak boleh 0.
EMT/Maxima akan memberi peringatan/error bila argumen di luar domain (lihat contoh pada file).

```
>plot2d("g(x)", -2, 6):
```



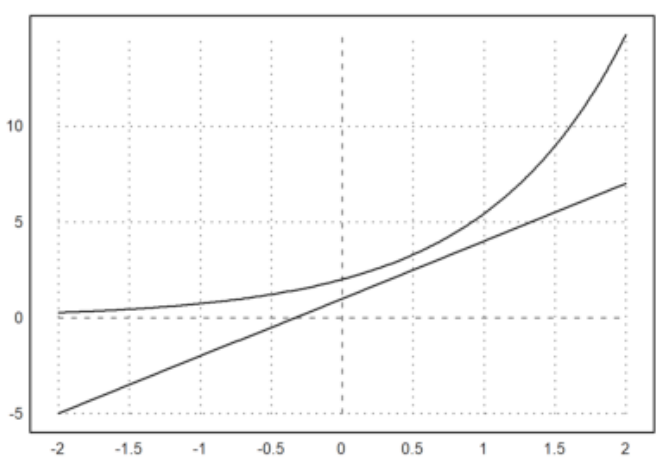

```
>function F(x) &= 2*E^x;
>function G(x) &= 3*x + 1;
>function H(x) &= F(G(x));
```

Fungsi trigonometrik dan komposisi (simbolik)

```
>H(0);
```

&= mendefinisikan fungsi simbolik (tergantung kebutuhan Anda — simbolik berguna jika ingin diff/integrate).

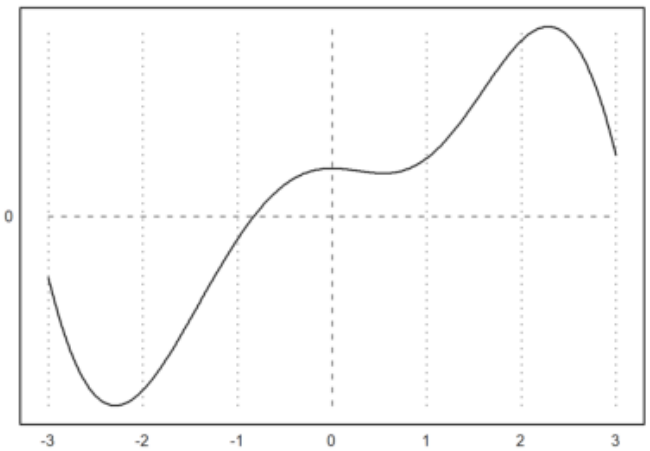
```
>plot2d(["F(x)","G(x)"], -2, 2):
```




```
>function s(x) := x^2*sin(x) + exp(-x^2);
>s(0); s(0:0.5:3.0);
```

Kombinasi ini sering muncul dalam soal aplikasi; exp(-x^2) menurun cepat sehingga plot berguna untuk memeriksa perilaku.

```
>plot2d("s(x)", -3, 3):
```



Menghitung Limit

Perhitungan limit pada EMT dapat dilakukan dengan menggunakan fungsi Maxima, yakni "limit". Fungsi "limit" dapat digunakan untuk menghitung limit fungsi dalam bentuk ekspresi maupun fungsi yang sudah didefinisikan sebelumnya. Nilai limit dapat dihitung pada sebarang nilai atau pada tak hingga (-inf, minf, dan inf). Limit kiri dan limit kanan juga dapat dihitung, dengan cara memberi opsi "plus" atau "minus". Hasil limit dapat berupa nilai, "und" (tak definisi), "ind" (tak tentu namun terbatas), "infinity" (kompleks tak hingga).

Perhatikan beberapa contoh berikut. Perhatikan cara menampilkan perhitungan secara lengkap, tidak hanya menampilkan hasilnya saja.

```
>$showev('limit(sqrt(x^2-3*x)/(x+1),x,inf))
```

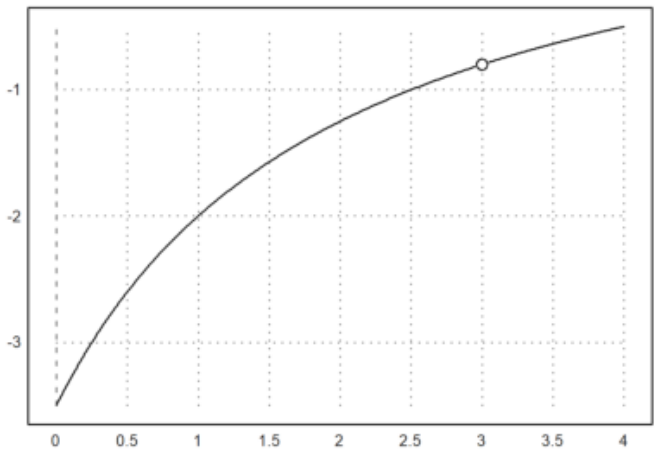
$$\lim_{x\to\infty} \frac{\sqrt{x^2-3\,x}}{x+1} = 1$$

```
>$limit((x^3-13*x^2+51*x-63)/(x^3-4*x^2-3*x+18),x,3)
```

$$-\frac{4}{5}$$

maxima: ’limit((x^3-13*x^2+51*x-63)/(x^3-4*x^2-3*x+18),x,3)=limit((x^3-13*x^2+51*x-63)/(x^3-4*x^2-3*x+18),x,3)
Fungsi tersebut diskontinu di titik x=3. Berikut adalah grafik fungsinya.

```
>aspect(1.5); plot2d("(x^3-13*x^2+51*x-63)/(x^3-4*x^2-3*x+18)",0,4); plot2d(3,-4/5,>points
```



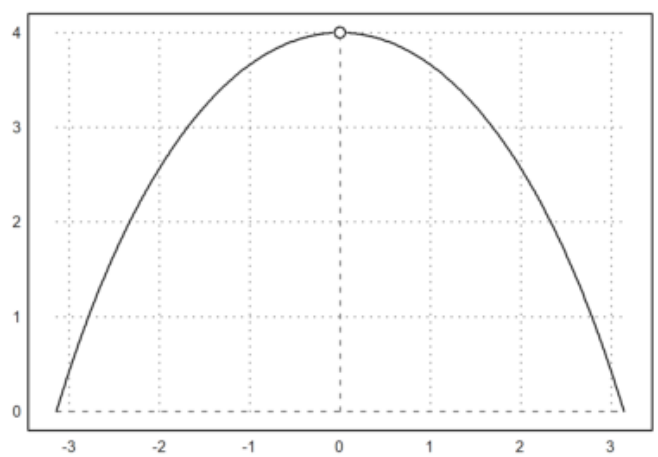

```
>$limit(2*x*sin(x)/(1-cos(x)),x,0)
```

4

$$2\left(\lim_{x\to 0}\frac{x\,\sin x}{1-\cos x}\right)=4$$

Fungsi tersebut diskontinu di titik x=0. Berikut adalah grafik fungsinya.

```
>plot2d("2*x*sin(x)/(1-cos(x))",-pi,pi); plot2d(0,4,>points,style="ow",>add):
```

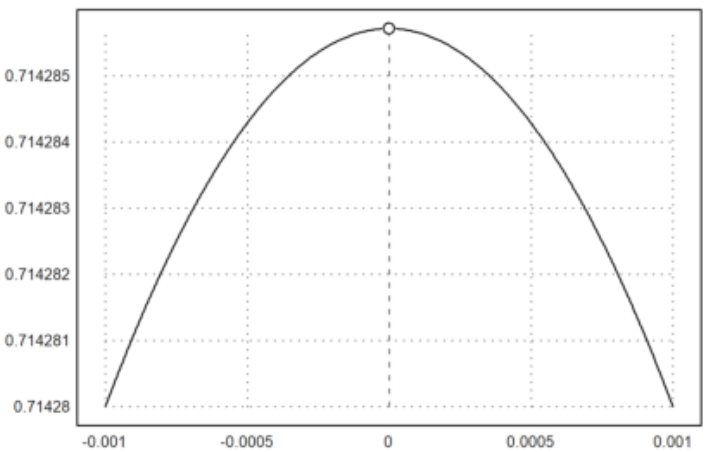


```
>$limit(cot(7*h)/cot(5*h),h,0)
```

$$\frac{5}{7}$$

maxima: showev('limit(cot(7*h)/cot(5*h),h,0))
Fungsi tersebut juga diskontinu (karena tidak terdefinisi) di x=0. Berikut adalah grafiknya.

```
>plot2d("cot(7*x)/cot(5*x)",-0.001,0.001); plot2d(0,5/7,>points,style="ow",>add):
```




```
>$showev('limit(((x/8)^(1/3)-1)/(x-8),x,8))
```

$$\lim_{x \to 8} \frac{\frac{x^{\frac{1}{3}}}{2} - 1}{x - 8} = \frac{1}{24}$$

Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

```
>$showev('limit(1/(2*x-1),x,0))
```

$$\lim_{x \to 0} \frac{1}{2\,x - 1} = -1$$

Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

```
>$showev('limit((x^2-3*x-10)/(x-5),x,5))
```

$$\lim_{x \to 5} \frac{x^2 - 3\,x - 10}{x - 5} = 7$$

Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

```
>$showev('limit(sqrt(x^2+x)-x,x,inf))
```

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \frac{1}{2}$$

Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

```
>$showev('limit(abs(x-1)/(x-1),x,1,minus))
```

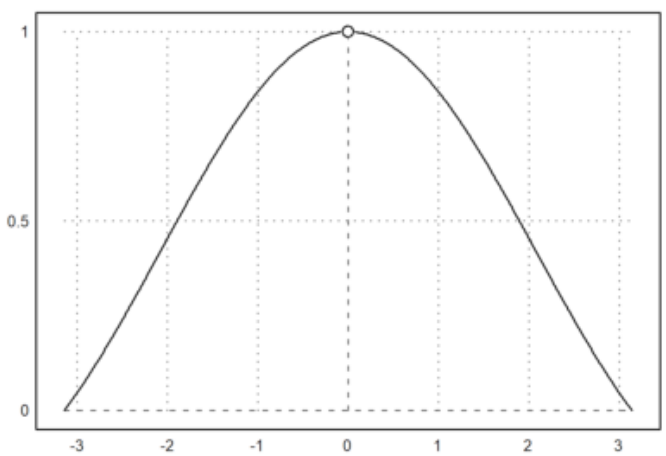
$$\lim_{x\uparrow 1} \frac{|x-1|}{x-1} = -1$$

Hitung limit di atas untuk x menuju 1 dari kanan.
Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

```
>$showev('limit(sin(x)/x,x,0))
```

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

```
>plot2d("sin(x)/x",-pi,pi); plot2d(0,1,>points,style="ow",>add):
```




```
>$showev('limit(sin(x^3)/x,x,0))
```

$$\lim_{x\to 0} \frac{\sin x^3}{x} = 0$$

Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

```
>$showev('limit(log(x), x, minf))
```

$$\lim_{x\to-\infty} \log x = infinity$$

```
>$showev('limit((-2)^x,x, inf))
```

$$\lim_{x\to\infty} (-2)^x = infinity$$

```
>$showev('limit(t-sqrt(2-t),t,2,minus))
```

$$\lim_{t\uparrow 2} t-\sqrt{2-t} = 2$$

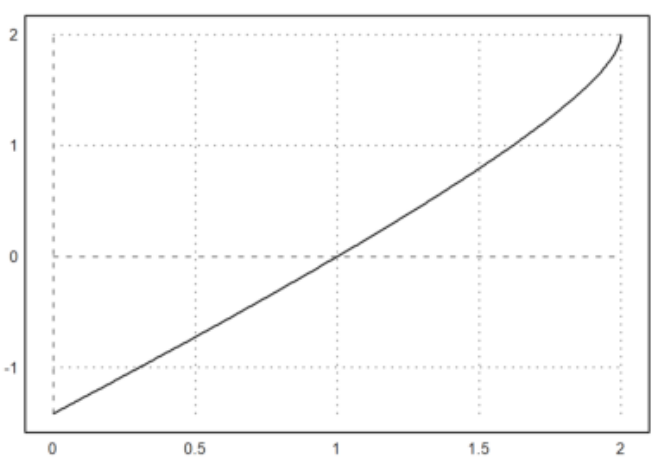
```
>$showev('limit(t-sqrt(2-t),t,2,plus))
```

$$\lim_{t\downarrow 2} t-\sqrt{2-t} = 2$$

```
>$showev('limit(t-sqrt(2-t),t,5,plus)) // Perhatikan hasilnya
```

$$\lim_{t\downarrow 5} t-\sqrt{2-t} = 5-\sqrt{3}\,i$$

```
>plot2d("x-sqrt(2-x)",0,2):
```



```
>$showev('limit((x^2-9)/(2*x^2-5*x-3),x,3))
```

$$\lim_{x\to 3} \frac{x^2-9}{2\,x^2-5\,x-3} = \frac{6}{7}$$

Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

```
>$showev('limit((1-cos(x))/x,x,0))
```

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

```
>$showev('limit((x^2+abs(x))/(x^2-abs(x)),x,0))
```

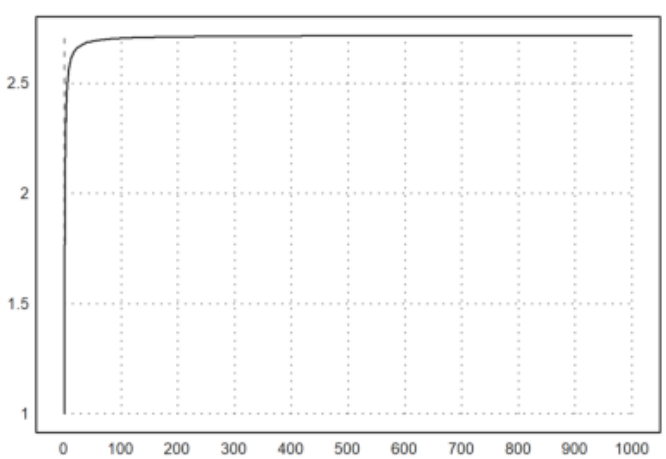
$$\lim_{x\to 0} \frac{|x|+x^2}{x^2-|x|} = -1$$

Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

```
>$showev('limit((1+1/x)^x,x,inf))
```

$$\lim_{x\to \infty} \left(\frac{1}{x}+1\right)^x = e$$

```
>plot2d("(1+1/x)^x",0,1000):
```



```
>$showev('limit((1+k/x)^x,x,inf))
```

$$\lim_{x\to\infty}\left(\frac{k}{x}+1\right)^{x}=e^{k}$$

```
>$showev('limit((1+x)^(1/x),x,0))
```

$$\lim_{x\to 0}(x+1)^{\frac{1}{x}}=e$$

Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

```
>$showev('limit((x/(x+k))^x,x,inf))
```

$$\lim_{x\to\infty}\left(\frac{x}{x+k}\right)^{x}=e^{-k}$$

```
>$showev('limit((E^x-E^2)/(x-2),x,2))
```

$$\lim_{x\to 2}\frac{e^{x}-e^{2}}{x-2}=e^{2}$$

Tunjukkan limit tersebut dengan grafik, seperti contoh-contoh sebelumnya.

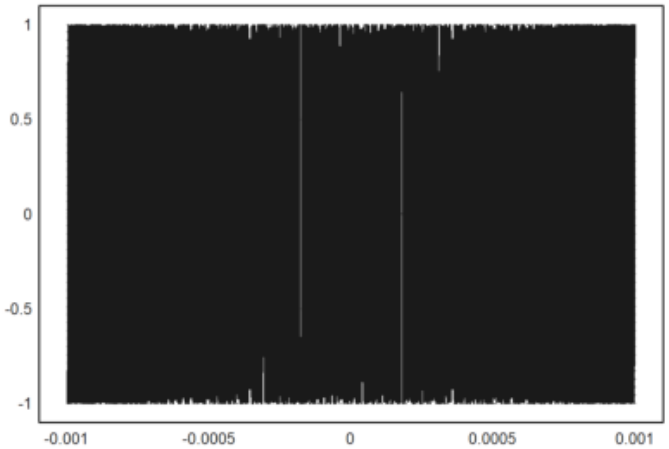
```
>$showev('limit(sin(1/x),x,0))
```

$$\lim_{x\to 0}\sin\left(\frac{1}{x}\right)=ind$$

```
>$showev('limit(sin(1/x),x,inf))
```

$$\lim_{x\to\infty}\sin\left(\frac{1}{x}\right)=0$$

```
>plot2d("sin(1/x)",-0.001,0.001):
```



Latihan

Bukalah buku Kalkulus. Cari dan pilih beberapa (paling sedikit 5 fungsi berbeda tipe/bentuk/jenis) fungsi dari buku tersebut, kemudian definisikan di EMT pada baris-baris perintah berikut (jika perlu tambahkan lagi). Untuk setiap fungsi, hitung nilai limit fungsi tersebut di beberapa nilai dan di tak hingga. Gambar grafik fungsi tersebut untuk mengkonfirmasi nilai-nilai limit tersebut.

Gunakan limit(expr, x, a) untuk limit biasa; limit(expr, x, a, plus) atau ..., minus untuk limit sebelah kanan/kiri.
Untuk limit di tak hingga gunakan limit(expr, x, inf) atau minf.
Bila Maxima kesulitan, pakai factor(...), ratsimp(...), atau substitusi aljabar sebelum limit. Contoh trik: faktor (E^(x+h)-E^x) ? (E^h-1)*E^x lalu ambil limit.
EMT4Kalkulus
Untuk verifikasi grafik gunakan plot2d("expr", xmin, xmax) dan tambahkan titik hasil dengan plot2d(x0,y0,>points,style="o

```
>$showev('limit(sin(x)/x, x, 0))
```

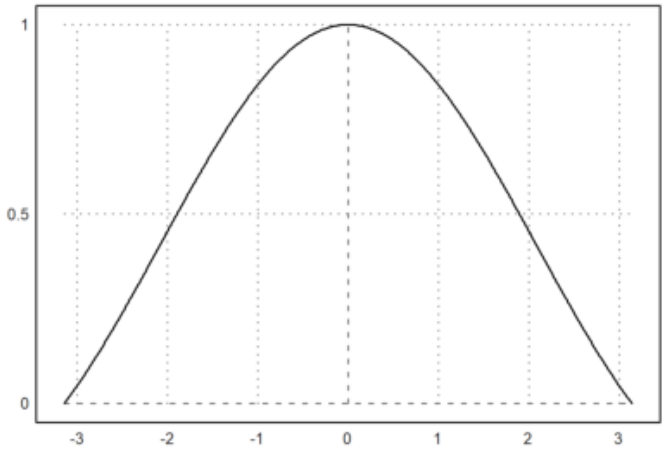
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

limit klasik: sin(x)/x saat x mendekati 0

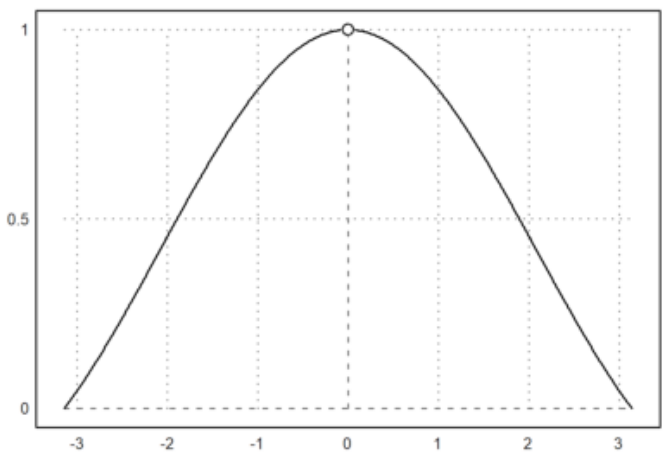
```
>/* verifikasi grafik */
```

```
  "skipcomment" illegal at this place!
  Variable verifikasi not found!
  Error in:
  /* verifikasi grafik */ ...
                ^
```

```
>plot2d("sin(x)/x", -pi, pi):
```




```
>plot2d(0,1, >points, style="ow", >add):
```



Hasil yang diharapkan: 1.
Alasan singkat: limit standar; dapat diperlihatkan juga dengan deret Taylor atau identitas geometri.

bentuk tak tentu yang perlu faktorisasi: (x^2-9)/(2*x^2-5*x-3) saat x mendekati 3

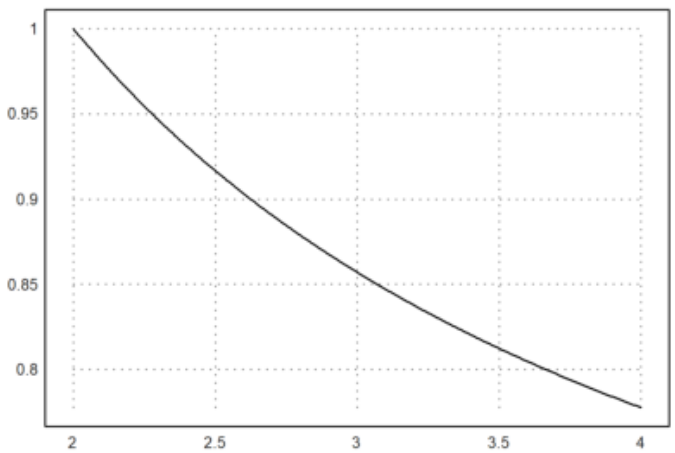
```
>$showev('limit((x^2-9)/(2*x^2-5*x-3), x, 3))
```

$$\lim_{x \to 3} \frac{x^2 - 9}{2\,x^2 - 5\,x - 3} = \frac{6}{7}$$

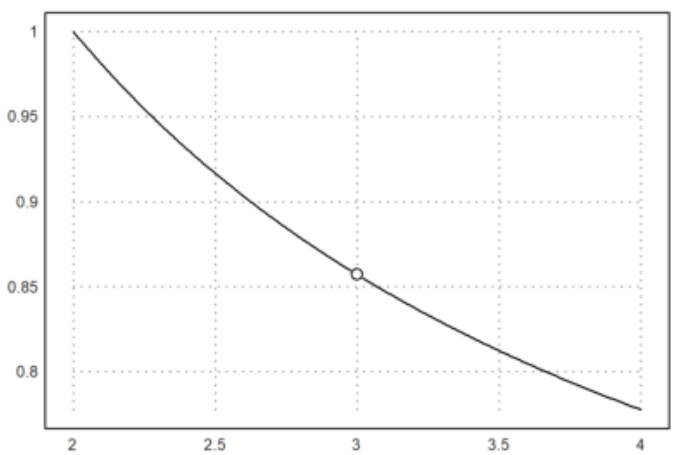
```
>/* gambarkan dan tandai titik */
```

```
"skipcomment" illegal at this place!
Variable gambarkan not found!
Error in:
/* gambarkan dan tandai titik */ ...
             ^
```

```
>plot2d("(x^2-9)/(2*x^2-5*x-3)", 2, 4):
```




```
>plot2d(3, 6/7, >points, style="ow", >add):
```



Hasil yang diharapkan: 6/7.
Trik: faktorkan pembilang/pemeriksa pembagi lalu sederhanakan sehingga tidak lagi 0/0 sebelum substitusi.

limit tak hingga yang menghasilkan e: (1 + 1/x)^x saat x mendekati tak hingga

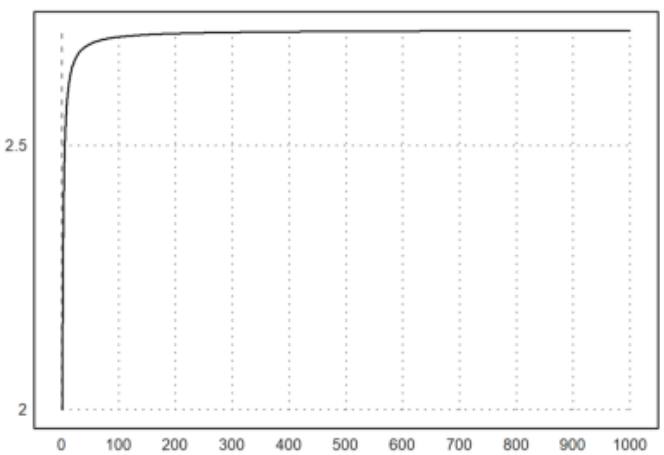
```
>$showev('limit((1 + 1/x)^x, x, inf))
```

$$\lim_{x\to\infty}\left(\frac{1}{x}+1\right)^{x}=e$$

```
>/* plot untuk konfirmasi numerik (plot domain besar) */
```

```
"skipcomment" illegal at this place!
Commands must be separated by semicolon or comma!
Found: untuk konfirmasi numerik (plot domain besar) */ (character 117)
You can disable this in the Options menu.
Error in:
/* plot untuk konfirmasi numerik (plot domain besar) */ ...
        ^
```

```
>plot2d("(1+1/x)^x", 1, 1000):
```



Hasil yang diharapkan: e.
Generalisasi: limit((1 + k/x)^x, x, inf) = exp(k)

limit satu-sisi dengan nilai mutlak: abs(x-1)/(x-1) saat x mendekati 1 (kiri/kanan)

```
>$showev('limit(abs(x-1)/(x-1), x, 1, minus))  /* x ? 1 (kiri) */
```

$$\lim_{x\uparrow 1} \frac{|x-1|}{x-1} = -1$$

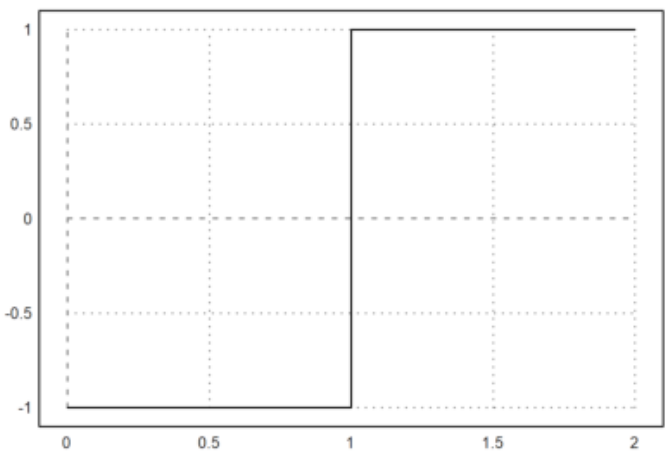
```
>$showev('limit(abs(x-1)/(x-1), x, 1, plus))   /* x ? 1 (kanan) */
```

$$\lim_{x\downarrow 1} \frac{|x-1|}{x-1} = 1$$

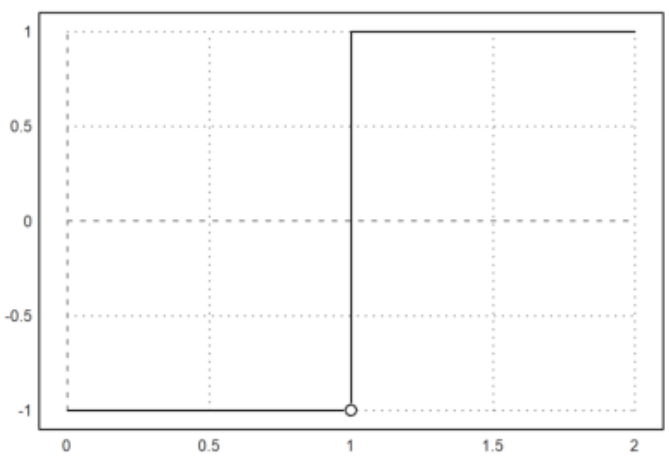
```
>/* Untuk melihat grafik (tampak diskontinuitas) */
```

```
"skipcomment" illegal at this place!
Variable Untuk not found!
Error in:
/* Untuk melihat grafik (tampak diskontinuitas) */ ...
         ^
```

```
>plot2d("abs(x-1)/(x-1)", 0, 2):
```




```
>plot2d(1, -1, >points, style="ow", >add): /* contoh titik limit kiri = -1 */
```




```
"skipcomment" illegal at this place!
Variable contoh not found!
Error in:
... t2d(1, -1, >points, style="ow", >add): /* contoh titik limit k ...
                                                    ^
```

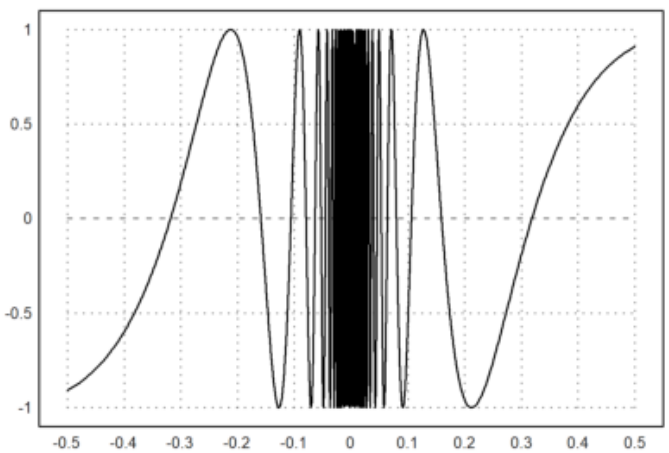
Hasil yang diharapkan: limit kiri = -1, limit kanan = 1.

limit osilasi/nonkonvergen: sin(1/x) saat x mendekati 0

```
>$showev('limit(sin(1/x), x, 0))
```

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = ind$$

```
>plot2d("sin(1/x)", -0.5, 0.5):
```



Cara menampilkan langkah perhitungan (bila ingin bukti kerja)

showev('limit(...)) atau $showev('limit(...)) menampilkan detail langkah perhitungan (lihat contoh di PDF). Untuk melihat hasil analitis lengkap gunakan showev(...); untuk hanya hasil singkat gunakan limit(...).

Trik bila Maxima mengeluh

(error/"floating point error" / "Is x an integer?")
Jika Maxima meminta asumsi (contoh: ketika menghitung limit yang bergantung pada tanda x), gunakan assume(x>0) atau assume(x<>0) sebelum memanggil limit, dan forget(...) setelah selesai. Contoh penggunaan assume untuk kasus x^x pada turunan ada di file.
EMT4Kalkulus
Jika limit memerlukan manipulasi aljabar, lakukan factor(expr) atau ratsimp(expr) lalu panggil limit pada ekspresi yang sudah disederhanakan.

Turunan Fungsi

Definisi turunan:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Berikut adalah contoh-contoh menentukan turunan fungsi dengan menggunakan definisi turunan (limit).

```
>$showev('limit(((x+h)^2-x^2)/h,h,0)) // turunan x^2
```

$$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=2\,x$$

```
>p &= expand((x+h)^2-x^2)|simplify; $p //pembilang dijabarkan dan disederhanakan
```

$$2\,h\,x+h^2$$

```
>q &=ratsimp(p/h); $q // ekspresi yang akan dihitung limitnya disederhanakan
```

$$2\,x+h$$

```
>$limit(q,h,0) // nilai limit sebagai turunan
```

$$2\,x$$

```
>$showev('limit(((x+h)^n-x^n)/h,h,0)) // turunan x^n
```

$$\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=n\,x^{n-1}$$

Mengapa hasilnya seperti itu? Tuliskan atau tunjukkan bahwa hasil limit tersebut benar, sehingga benar turunan fungsinya benar. Tulis penjelasan Anda di komentar ini.
Sebagai petunjuk, ekspansikan (x+h)^n dengan menggunakan teorema binomial.

```
>$showev('limit((sin(x+h)-sin(x))/h,h,0)) // turunan sin(x)
```

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=\cos x$$

Mengapa hasilnya seperti itu? Tuliskan atau tunjukkan bahwa hasil limit tersebut benar, sehingga benar turunan fungsinya benar. Tulis penjelasan Anda di komentar ini.
Sebagai petunjuk, ekspansikan sin(x+h) dengan menggunakan rumus jumlah dua sudut.

```
>$showev('limit((log(x+h)-log(x))/h,h,0)) // turunan log(x)
```

$$\lim_{h\to 0} \frac{\log(x+h) - \log x}{h} = \frac{1}{x}$$

Mengapa hasilnya seperti itu? Tuliskan atau tunjukkan bahwa hasil limit tersebut benar, sehingga benar turunan fungsinya benar. Tulis penjelasan Anda di komentar ini.
Sebagai petunjuk, gunakan sifat-sifat logaritma dan hasil limit pada bagian sebelumnya di atas.

```
>$showev('limit((1/(x+h)-1/x)/h,h,0)) // turunan 1/x
```

$$\lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = -\frac{1}{x^2}$$

```
>$showev('limit((E^(x+h)-E^x)/h,h,0)) // turunan f(x)=e^x
```

```
Answering "Is x an integer?" with "integer"
Answering "Is x an integer?" with "integer"
Answering "Is x an integer?" with "integer"
Answering "Is x an integer?" with "integer"
Answering "Is x an integer?" with "integer"
Maxima is asking
Acceptable answers are: yes, y, Y, no, n, N, unknown, uk
Is x an integer?

Use assume!
Error in:
$showev('limit((E^(x+h)-E^x)/h,h,0)) // turunan f(x)=e^x ...
                                     ^
```

Maxima bermasalah dengan limit:

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h}.$$

Oleh karena itu diperlukan trik khusus agar hasilnya benar.

```
>$showev('limit((E^h-1)/h,h,0))
```

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

```
>$showev('factor(E^(x+h)-E^x))
```

$$factor\left(e^{x+h}-e^{x}\right)=\left(e^{h}-1\right)\,e^{x}$$

```
>$showev('limit(factor((E^(x+h)-E^x)/h),h,0)) // turunan f(x)=e^x
```

$$\left(\lim_{h\to 0}\frac{e^{h}-1}{h}\right)\,e^{x}=e^{x}$$

```
>function f(x) &= x^x
```

$$\begin{matrix}\mathrm{x}\\ \mathrm{x}\end{matrix}$$

```
>$showev('limit(f(x),x,0))
```

$$\lim_{x\to 0}x^{x}=1$$

Silakan Anda gambar kurva

$$y=x^{x}.$$

```
>$showev('limit((f(x+h)-f(x))/h,h,0)) // turunan f(x)=x^x
```

$$\lim_{h\to 0}\frac{(x+h)^{x+h}-x^{x}}{h}=infinity$$

Di sini Maxima juga bermasalah terkait limit:

$$\lim_{h\to 0}\frac{(x+h)^{x+h}-x^{x}}{h}.$$

Dalam hal ini diperlukan asumsi nilai x.

```
>&assume(x>0); $showev('limit((f(x+h)-f(x))/h,h,0)) // turunan f(x)=x^x
```

$$\lim_{h\to 0}\frac{(x+h)^{x+h}-x^{x}}{h}=x^{x}\,\left(\log x+1\right)$$

Mengapa hasilnya seperti itu? Tuliskan atau tunjukkan bahwa hasil limit tersebut benar, sehingga benar turunan fungsinya benar. Tulis penjelasan Anda di komentar ini.

```
>&forget(x>0) // jangan lupa, lupakan asumsi untuk kembali ke semula
```

[x > 0]

```
>&forget(x<0)
```

[x < 0]

```
>&facts()
```

[]

```
>$showev('limit((asin(x+h)-asin(x))/h,h,0)) // turunan arcsin(x)
```

$$\lim_{h\to 0} \frac{\arcsin(x+h)-\arcsin x}{h} = \frac{1}{\sqrt{1-x^2}}$$

Mengapa hasilnya seperti itu? Tuliskan atau tunjukkan bahwa hasil limit tersebut benar, sehingga benar turunan fungsinya benar. Tulis penjelasan Anda di komentar ini.

```
>$showev('limit((tan(x+h)-tan(x))/h,h,0)) // turunan tan(x)
```

$$\lim_{h\to 0} \frac{\tan(x+h)-\tan x}{h} = \frac{1}{\cos^2 x}$$

Mengapa hasilnya seperti itu? Tuliskan atau tunjukkan bahwa hasil limit tersebut benar, sehingga benar turunan fungsinya benar. Tulis penjelasan Anda di komentar ini.

```
>function f(x) &= sinh(x) // definisikan f(x)=sinh(x)
```

sinh(x)

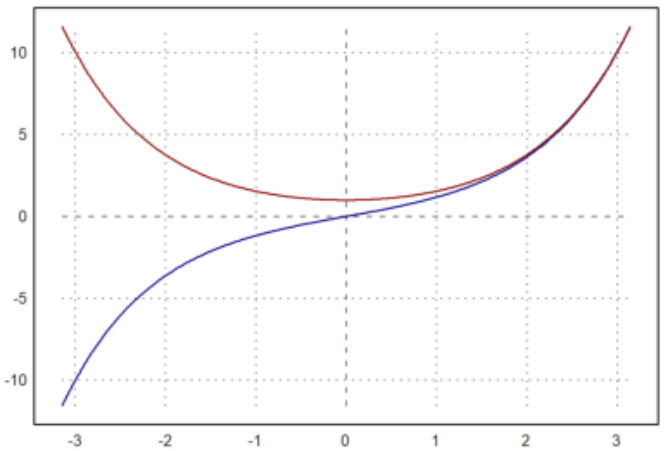
```
>function df(x) &= limit((f(x+h)-f(x))/h,h,0); $df(x) // df(x) = f'(x)
```

$$\frac{e^{-x}\left(e^{2x}+1\right)}{2}$$

Hasilnya adalah cosh(x), karena

$$\frac{e^x + e^{-x}}{2} = \cosh(x).$$

```
>plot2d(["f(x)","df(x)"],-pi,pi,color=[blue,red]):
```




```
>function f(x) &= sin(3*x^5+7)^2
```

$$\sin\left(3\,x^5+7\right)^2$$

```
>diff(f,3), diffc(f,3)
```

```
  1198.32948904
  1198.72863721
```

Apakah perbedaan diff dan diffc?

```
>$showev('diff(f(x),x))
```

$$\frac{d}{d\,x}\,\sin^2\left(3\,x^5+7\right) = 30\,x^4\,\cos\left(3\,x^5+7\right)\,\sin\left(3\,x^5+7\right)$$

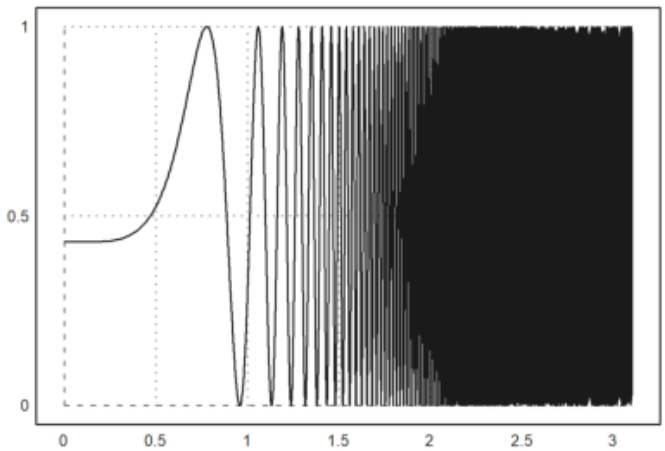
```
>$% with x=3
```

$$\%at\left(\frac{d}{d\,x}\,\sin^2\left(3\,x^5+7\right), x=3\right) = 2430\,\cos 736\,\sin 736$$

```
>$float(%)
```

$$\%at\left(\frac{d^{1.0}}{d\,x^{1.0}}\,\sin^2\left(3.0\,x^5+7.0\right),x=3.0\right)=1198.728637211748$$

```
>plot2d(f,0,3.1):
```




```
>function f(x) &=5*cos(2*x)-2*x*sin(2*x) // mendifinisikan fungsi f
```

$$5\ \cos(2\ x)\ -\ 2\ x\ \sin(2\ x)$$

```
>function df(x) &=diff(f(x),x) // fd(x) = f'(x)
```

$$-\ 12\ \sin(2\ x)\ -\ 4\ x\ \cos(2\ x)$$

```
>$'f(1)=f(1), $float(f(1)), $'f(2)=f(2), $float(f(2)) // nilai f(1) dan f(2)
```

$$f\left(1\right)=5\,\cos 2-2\,\sin 2$$

$$-3.899329036387075$$

$$f\left(2\right)=5\,\cos 4-4\,\sin 4$$

$$-0.2410081230863468$$

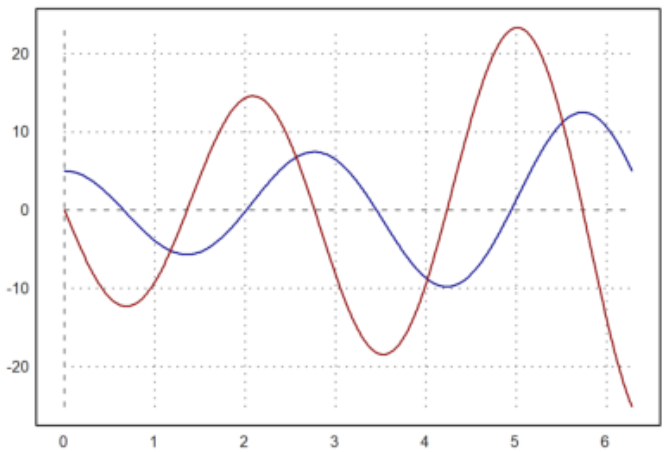
```
>xp=solve("df(x)",1,2,0) // solusi f'(x)=0 pada interval [1, 2]
```

```
  1.35822987384
```

```
>df(xp), f(xp) // cek bahwa f'(xp)=0 dan nilai ekstrim di titik tersebut
```

```
  0
  -5.67530133759
```

```
>plot2d(["f(x)","df(x)"],0,2*pi,color=[blue,red]): //grafik fungsi dan turunannya
```



Perhatikan titik-titik "puncak" grafik y=f(x) dan nilai turunan pada saat grafik fungsinya mencapai titik "puncak" tersebut.

Latihan

Bukalah buku Kalkulus. Cari dan pilih beberapa (paling sedikit 5 fungsi berbeda tipe/bentuk/jenis) fungsi dari buku tersebut, kemudian definisikan di EMT pada baris-baris perintah berikut (jika perlu tambahkan lagi). Untuk setiap fungsi, tentukan turunannya dengan menggunakan definisi turunan (limit), menggunakan perintah diff, dan secara manual (langkah demi langkah yang dihitung dengan Maxima) seperti contoh-contoh di atas. Gambar grafik fungsi asli dan fungsi turunannya pada sumbu koordinat yang sama.

Turunan polinomial (pakai definisi limit)

$$f(x) = x^n$$

(turunan = nx^(n-1))

```
>/* definisi simbolik */
```

```
  "skipcomment" illegal at this place!
  Variable definisi not found!
  Error in:
  /* definisi simbolik */ ...
             ^
```

```
>$n:3 /* ubah n bila perlu */
```

$$3$$

```
>maxima $f(x) := x^n
```

$$f(x) := x^3$$

```
>$showev('limit(((x+h)^n - x^n)/h, h, 0))
```

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = 3\,x^2$$

```
>/* atau secara langsung diff */
```

```
"skipcomment" illegal at this place!
Variable atau not found!
Error in:
/* atau secara langsung diff */ ...
        ^
```

```
>$diff(f(x), x)
```

$$3\,x^2$$

Hasil yang diharapkan: untuk n=3 adalah

$$3x^2$$

Penjelasan: perlu ekspansi binomial pada langkah limit; Maxima menampilkan langkah bila pakai showev.

Turunan sin(x) lewat definisi

$$f(x) = sinx$$

```
>$showev('limit((sin(x+h)-sin(x))/h, h, 0))
```

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$$

```
>/* atau */
>maxima f(x) := sin(x)
```

$$f(x) := \sin(x)$$

```
>$diff(f(x), x)
```

$$\cos x$$

Hasil: cos x.
Catatan: gunakan identitas jumlah sudut untuk pembuktian limit; showev menampilkan rangkaian aljabarnya.

Turunan log(x) lewat definisi

$$f(x) = logx$$

```
>$showev('limit((log(x+h)-log(x))/h, h, 0))
```

$$\lim_{h \to 0} \frac{\log(x+h) - \log x}{h} = \frac{1}{x}$$

```
>/* atau */
```

```
  "skipcomment" illegal at this place!
  Syntax error in expression, or unfinished expression!
  Error in:
  /* atau */ ...
           ^
```

```
>maxima f(x) := log(x)
```

$$f(x) := \log(x)$$

```
>$diff(f(x), x)
```

$$\frac{1}{x}$$

Hasil : 1/x
Trik: gunakan sifat logaritma untuk menyederhanakan ekspresi sebelum mengambil limit

```
>/* atau */
```

Turunan fungsi komplikasi

$$f(x) = x^x$$

```
>/* gunakan asumsi x>0 lalu hitung turunan simbolik */
```

```
  "skipcomment" illegal at this place!
  Variable gunakan not found!
  Error in:
  /* gunakan asumsi x>0 lalu hitung turunan simbolik */ ...
             ^
```

```
>maxima assume(x>0)
```

```
[x > 0]
```

```
>maxima f(x)= x^x
```

```
                    x
log(x) = x
```

```
>$showev('limit((f(x+h)-f(x))/h, h, 0)) /* definisi: kadang Maxima kesulitan */
```

$$\lim_{h \to 0} \frac{\log(x+h) - \log x}{h} = \frac{1}{x}$$

```
>maxima diff(f(x), x) /* Maxima biasanya bisa menghasilkan turunan simbolik: x^x*(log(x)+1
```

```
1
-
x
```

```
>maxima forget(x>0)
```

```
[x > 0]
```

Hasil: x^x (ln x +1) (dengan asumsi x>0)
Catatan: Maxima memerlukan assume(x>0) agar limit definisi dapat dihitung

Turunan trig lebih rumit & plotting fungsi + turunan

$$f(x) = 5cos(2x) - 2xsin(2x)$$

```
>maxima function f(x) := 5*cos(2*x) - 2*x*sin(2*x)
```

```
f(x) := 5 cos(2 x) - 2 x sin(2 x)
```

```
>/* simbolik turunan */
```

```
"skipcomment" illegal at this place!
Variable simbolik not found!
Error in:
/* simbolik turunan */ ...
            ^
```

```
>function df(x) &= diff(f(x), x)
```

```
- 12 sin(2 x) - 4 x cos(2 x)
```

```
>maxima f(1), float(f(1))
```

```
5 cos(2) - 2 sin(2)
```

```
- 3.899329036387075
```

```
>maxima df(1), float(df(1))
```

```
- 12 sin(2) - 4 cos(2)

- 9.24698177571961
```

```
>/* mencari akar turunan pada interval [1,2] */)
```

```
  "skipcomment" illegal at this place!
  Variable mencari not found!
  Error in:
  /* mencari akar turunan pada interval [1,2] */) ...
             ^
```

```
>maxima xp = solve("df(x)", 1, 2, 0)  /* solusi f'(x)=0 pada interval */
```

```
  Maxima encountered a Lisp error:

   $SOLVE called with invalid number of arguments: 4

  Automatically continuing.
  To enable the Lisp debugger set *debugger-hook* to nil.
```

```
>maxima df(xp), f(xp)
```

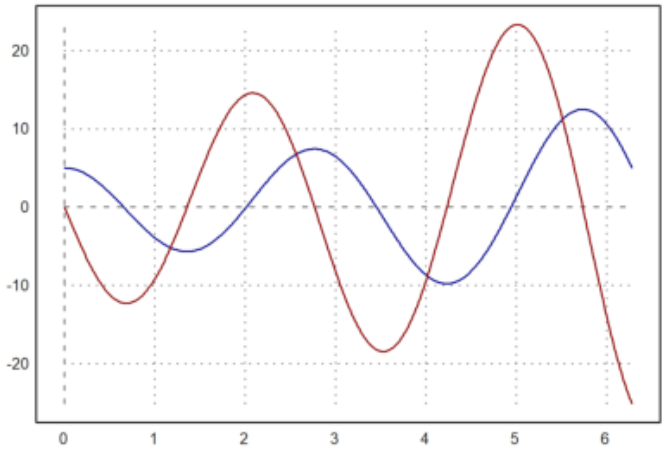
```
                    - 12 sin(2 xp) - 4 xp cos(2 xp)

                      5 cos(2 xp) - 2 xp sin(2 xp)
```

```
>/* plot fungsi & turunannya bersama */
```

```
  "skipcomment" illegal at this place!
  Commands must be separated by semicolon or comma!
  Found: fungsi & turunannya bersama */ (character 102)
  You can disable this in the Options menu.
  Error in:
  /* plot fungsi & turunannya bersama */ ...
          ^
```

```
>plot2d(["f(x)","df(x)"], 0, 2*pi, color=[blue,red]):
```



Integral

EMT dapat digunakan untuk menghitung integral, baik integral tak tentu maupun integral tentu. Untuk integral tak tentu (simbolik) sudah tentu EMT menggunakan Maxima, sedangkan untuk perhitungan integral tentu EMT sudah menyediakan beberapa fungsi yang mengimplementasikan algoritma kuadratur (perhitungan integral tentu menggunakan metode numerik).

Pada notebook ini akan ditunjukkan perhitungan integral tentu dengan menggunakan Teorema Dasar Kalkulus:

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{dengan } F'(x) = f(x).$$

Fungsi untuk menentukan integral adalah integrate. Fungsi ini dapat digunakan untuk menentukan, baik integral tentu maupun tak tentu (jika fungsinya memiliki antiderivatif). Untuk perhitungan integral tentu fungsi integrate menggunakan metode numerik (kecuali fungsinya tidak integrabel, kita tidak akan menggunakan metode ini).

```
>$showev('integrate(x^n,x))
```

$$\int x^3\,dx = \frac{x^4}{4}$$

```
>$showev('integrate(1/(1+x),x))
```

$$\int \frac{1}{x+1}\,dx = \log(x+1)$$

```
>$showev('integrate(1/(1+x^2),x))
```

$$\int \frac{1}{x^2+1}\,dx = \arctan x$$

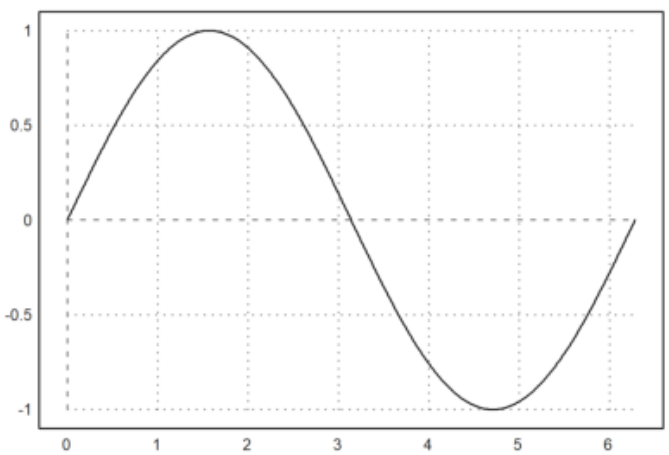
```
>$showev('integrate(1/sqrt(1-x^2),x))
```

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x$$

```
>$showev('integrate(sin(x),x,0,pi))
```

$$\int_0^{\pi} \sin x\,dx = 2$$

```
>plot2d("sin(x)",0,2*pi):
```




```
>$showev('integrate(sin(x),x,a,b))
```

$$\int_a^b \sin x\,dx = \cos a - \cos b$$

```
>$showev('integrate(x^n,x,a,b))
```

$$\int_a^b x^3\,dx = \frac{b^4}{4} - \frac{a^4}{4}$$

```
>$showev('integrate(x^2*sqrt(2*x+1),x))
```

$$\int x^2\sqrt{2\,x+1}\,dx = \frac{(2\,x+1)^{\frac{7}{2}}}{28} - \frac{(2\,x+1)^{\frac{5}{2}}}{10} + \frac{(2\,x+1)^{\frac{3}{2}}}{12}$$

```
>$showev('integrate(x^2*sqrt(2*x+1),x,0,2))
```

$$\int_0^2 x^2 \sqrt{2\,x+1}\,dx = \frac{2\,5^{\frac{5}{2}}}{21} - \frac{2}{105}$$

```
>$ratsimp(%)
```

$$\int_0^2 x^2 \sqrt{2\,x+1}\,dx = \frac{2\,5^{\frac{7}{2}} - 2}{105}$$

```
>$showev('integrate((sin(sqrt(x)+a)*E^sqrt(x))/sqrt(x),x,0,pi^2))
```

$$\int_0^{\pi^2} \frac{\sin\left(\sqrt{x}+a\right)\,e^{\sqrt{x}}}{\sqrt{x}}\,dx = \left(-e^{\pi}-1\right)\,\sin a + \left(e^{\pi}+1\right)\,\cos a$$

```
>$factor(%)
```

$$\int_0^{\pi^2} \frac{\sin\left(\sqrt{x}+a\right)\,e^{\sqrt{x}}}{\sqrt{x}}\,dx = \left(-e^{\pi}-1\right)\,\left(\sin a - \cos a\right)$$

```
>function map f(x) &= E^(-x^2)
```

$$\begin{matrix} & 2 \\ - & x \\ E & \end{matrix}$$

```
>$showev('integrate(f(x),x))
```

$$\int e^{-x^2}\,dx = \frac{\sqrt{\pi}\,\mathrm{erf}\left(x\right)}{2}$$

Fungsi f tidak memiliki antiturunan, integralnya masih memuat integral lain.

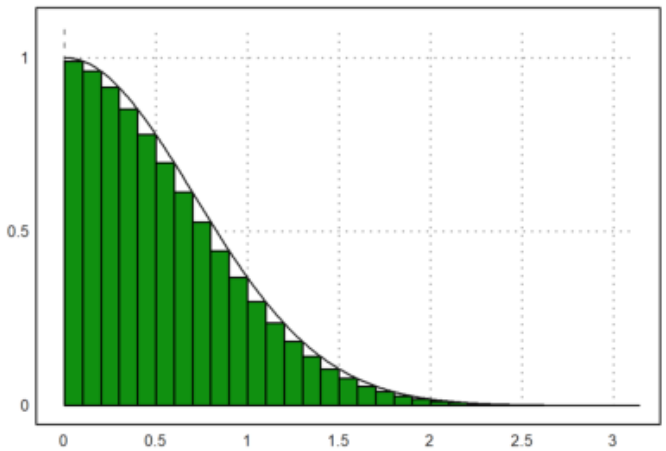
$$erf(x) = \int \frac{e^{-x^2}}{\sqrt{\pi}}\,dx.$$

Kita tidak dapat menggunakan teorema Dasar kalkulus untuk menghitung integral tentu fungsi tersebut jika semua batasnya berhingga. Dalam hal ini dapat digunakan metode numerik (rumus kuadratur).

Misalkan kita akan menghitung:

maxima: 'integrate(f(x),x,0,pi)

```
>x=0:0.1:pi-0.1; plot2d(x,f(x+0.1),>bar); plot2d("f(x)",0,pi,>add):
```



Integral tentu

maxima: 'integrate(f(x),x,0,pi)

dapat dihampiri dengan jumlah luas persegi-persegi panjang di bawah kurva y=f(x) tersebut. Langkah-langkahnya adalah sebagai berikut.

```
>t &= makelist(a,a,0,pi-0.1,0.1); // t sebagai list untuk menyimpan nilai-nilai x
>fx &= makelist(f(t[i]+0.1),i,1,length(t)); // simpan nilai-nilai f(x)
>// jangan menggunakan x sebagai list, kecuali Anda pakar Maxima!
```

Hasilnya adalah:

maxima: 'integrate(f(x),x,0,pi) = 0.1*sum(fx[i],i,1,length(fx))

Jumlah tersebut diperoleh dari hasil kali lebar sub-subinterval (=0.1) dan jumlah nilai-nilai f(x) untuk x = 0.1, 0.2, 0.3, ..., 3.2.

```
>0.1*sum(f(x+0.1)) // cek langsung dengan perhitungan numerik EMT
```

```
  0.836219610253
```

Untuk mendapatkan nilai integral tentu yang mendekati nilai sebenarnya, lebar sub-intervalnya dapat diperkecil lagi, sehingga daerah di bawah kurva tertutup semuanya, misalnya dapat digunakan lebar subinterval 0.001. (Silakan dicoba!)

Meskipun Maxima tidak dapat menghitung integral tentu fungsi tersebut untuk batas-batas yang berhingga, namun integral tersebut dapat dihitung secara eksak jika batas-batasnya tak hingga. Ini adalah salah satu keajaiban di dalam matematika, yang terbatas tidak dapat dihitung secara eksak, namun yang tak hingga malah dapat dihitung secara eksak.

```
>$showev('integrate(f(x),x,0,inf))
```

$$\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Tunjukkan kebenaran hasil di atas!

Berikut adalah contoh lain fungsi yang tidak memiliki antiderivatif, sehingga integral tentunya hanya dapat dihitung dengan metode numerik.

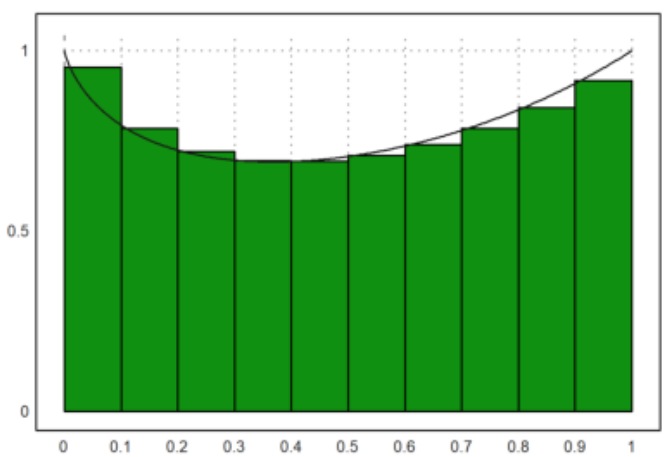
```
>function f(x) &= x^x
```

$$x$$
$$x$$

```
>$showev('integrate(f(x),x,0,1))
```

$$\int_0^1 x^x\,dx = \int_0^1 x^x\,dx$$

```
>x=0:0.1:1-0.01; plot2d(x,f(x+0.01),>bar); plot2d("f(x)",0,1,>add):
```



Maxima gagal menghitung integral tentu tersebut secara langsung menggunakan perintah integrate. Berikut kita lakukan seperti contoh sebelumnya untuk mendapat hasil atau pendekatan nilai integral tentu tersebut.

```
>t &= makelist(a,a,0,1-0.01,0.01);
>fx &= makelist(f(t[i]+0.01),i,1,length(t));
```

maxima: 'integrate(f(x),x,0,1) = 0.01*sum(fx[i],i,1,length(fx))
Apakah hasil tersebut cukup baik? perhatikan gambarnya.

```
>function f(x) &= sin(3*x^5+7)^2
```

$$\sin\left(3\,x^5+7\right)^2$$

```
>integrate(f,0,1)
```

```
  0.542581176074
```

```
>&showev('integrate(f(x),x,0,1))
```

$$\int_0^1 \sin\left(3\,x^5+7\right)^2\,dx = \frac{\Gamma\left(\frac{1}{5}\right)\sin(14)\,\sin\left(\frac{\pi}{10}\right)}{10^{1/5}\,6}$$

$$-\left(\left(6\,\Gamma\left(\frac{4}{5},6\,I\right)+6\,\Gamma\left(\frac{4}{5},-6\,I\right)\right)\sin(14)+\left(6\,I\,\Gamma\left(\frac{4}{5},6\,I\right)-6\,I\,\Gamma\left(\frac{4}{5},-6\,I\right)\right)\cos(14)\right)\sin\left(\frac{\pi}{10}\right)-60\Big)/120$$

```
>&float(%)
```

$$\int_0^{1.0} \sin\left(3.0\,x^5+7.0\right)\,dx =$$

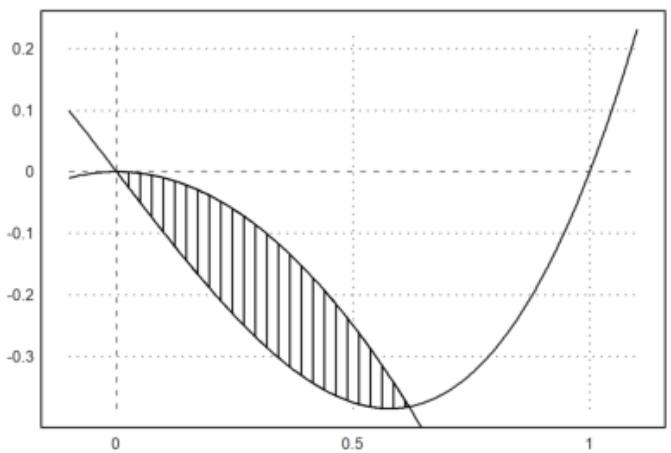
$$0.09820784258795788-0.008333333333333333\,(0.3090169943749474\,(0.1367372182078336\,(4.192962712629476\,I\,\Gamma(0.2,6.0\,I)-4.192962712629476\,I\,\Gamma(0.2,-6.0\,I))+0.9906073556948704\,(4.192962712629476\,\Gamma(0.2,6.0\,I)+4.192962712629476\,\Gamma(0.2,-6.0\,I)))-60.0)$$

```
>$showev('integrate(x*exp(-x),x,0,1)) // Integral tentu (eksak)
```

$$\int_0^1 x\,e^{-x}\,dx = 1 - 2\,e^{-1}$$

Aplikasi Integral Tentu

```
>plot2d("x^3-x",-0.1,1.1); plot2d("-x^2",>add);  ...
>b=solve("x^3-x+x^2",0.5); x=linspace(0,b,200); xi=flipx(x); ...
>plot2d(x|xi,x^3-x|-xi^2,>filled,style="|",fillcolor=1,>add): // Plot daerah antara 2 kurv
```




```
>a=solve("x^3-x+x^2",0), b=solve("x^3-x+x^2",1) // absis titik-titik potong kedua kurva
```

```
  0
  0.61803398875
```

```
>integrate("(-x^2)-(x^3-x)",a,b) // luas daerah yang diarsir
```

```
  0.0758191713542
```

Hasil tersebut akan kita bandingkan dengan perhitungan secara analitik.

```
>a &= solve((-x^2)-(x^3-x),x); $a // menentukan absis titik potong kedua kurva secara eksa
```

$$\left[x = \frac{-\sqrt{5}-1}{2}, x = \frac{\sqrt{5}-1}{2}, x = 0\right]$$

```
>$showev('integrate(-x^2-x^3+x,x,0,(sqrt(5)-1)/2)) // Nilai integral secara eksak
```

$$\int_0^{\frac{\sqrt{5}-1}{2}} -x^3 - x^2 + x \, dx = \frac{13 - 5^{\frac{3}{2}}}{24}$$

```
>$float(%)
```

$$\int_{0.0}^{0.6180339887498949} -1.0\,x^3 - 1.0\,x^2 + x \, dx = 0.07581917135421037$$

Panjang Kurva

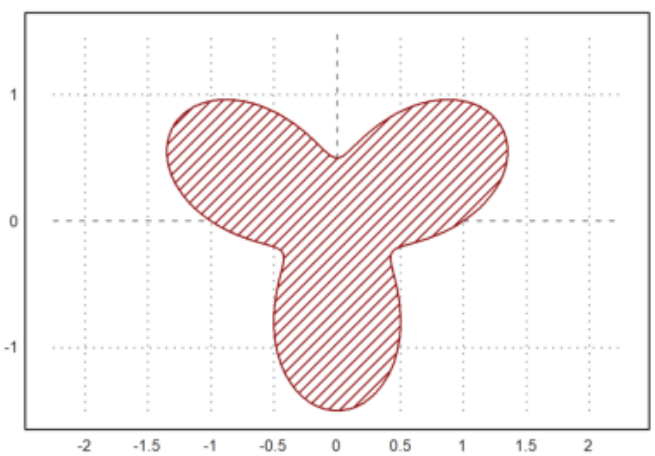
Hitunglah panjang kurva berikut ini dan luas daerah di dalam kurva tersebut.

$$\gamma(t) = (r(t)\cos(t), r(t)\sin(t))$$

dengan

$$r(t) = 1 + \frac{\sin(3t)}{2}, \quad 0 \le t \le 2\pi.$$

```
>t=linspace(0,2pi,1000); r=1+sin(3*t)/2; x=r*cos(t); y=r*sin(t); ...
>plot2d(x,y,>filled,fillcolor=red,style="/",r=1.5): // Kita gambar kurvanya terlebih dahul
```




```
>maxima function r(t) &= 1+sin(3*t)/2; $'r(t)=r(t)
```

```
        r([0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09,
0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21,
0.2200000000000001, 0.2300000000000001, 0.2400000000000001,
0.2500000000000001, 0.2600000000000001, 0.2700000000000001,
0.2800000000000001, 0.2900000000000001, 0.3000000000000001,
0.3100000000000001, 0.3200000000000001, 0.3300000000000001,
0.3400000000000001, 0.3500000000000001, 0.3600000000000002,
```

```
0.3700000000000002, 0.3800000000000002, 0.3900000000000002,
0.4000000000000002, 0.4100000000000002, 0.4200000000000002,
0.4300000000000002, 0.4400000000000002, 0.4500000000000002,
0.4600000000000002, 0.4700000000000003, 0.4800000000000003,
0.4900000000000003, 0.5000000000000002, 0.5100000000000002,
0.5200000000000002, 0.5300000000000002, 0.5400000000000003,
0.5500000000000003, 0.5600000000000003, 0.5700000000000003,
0.5800000000000003, 0.5900000000000003, 0.6000000000000003,
0.6100000000000003, 0.6200000000000003, 0.6300000000000003,
0.6400000000000003, 0.6500000000000004, 0.6600000000000004,
0.6700000000000004, 0.6800000000000004, 0.6900000000000004,
0.7000000000000004, 0.7100000000000004, 0.7200000000000004,
0.7300000000000004, 0.7400000000000004, 0.7500000000000004,
0.7600000000000005, 0.7700000000000005, 0.7800000000000005,
0.7900000000000005, 0.8000000000000005, 0.8100000000000005,
0.8200000000000005, 0.8300000000000005, 0.8400000000000005,
0.8500000000000005, 0.8600000000000005, 0.8700000000000006,
0.8800000000000006, 0.8900000000000006, 0.9000000000000006,
0.9100000000000006, 0.9200000000000006, 0.9300000000000006,
0.9400000000000006, 0.9500000000000006, 0.9600000000000006,
0.9700000000000006, 0.9800000000000006, 0.9900000000000007]) =
[1, 1.014997750101248, 1.029982003239722, 1.044939274599006,
1.05985610364446, 1.0747190662368, 1.089514786712912,
1.10422994992305, 1.118851313213567, 1.133365718344415,
1.14776010333067, 1.162021514197434, 1.176137116637545,
1.190094207561581, 1.203880226529785, 1.217482767055615,
1.230889587770742, 1.244088623441454, 1.257067995826556,
1.269816024366985, 1.282321236697518, 1.294572378971135,
1.306558425986717, 1.318268591110984, 1.329692335985737,
1.340819380011667, 1.351639709600205, 1.362143587185071,
1.37232155998543, 1.382164468512753, 1.391663454813742,
1.400809970441889, 1.409595784150499, 1.41801298930026,
1.426054010974682, 1.433711612797009, 1.440978903442474,
1.447849342840024, 1.454316748057942, 1.460375298868068,
1.466019542983613, 1.471244400965849, 1.476045170795258,
1.480417532103036, 1.484357550059133, 1.48786167891333,
1.49092676518618, 1.493550050506925, 1.495729174095843,
1.49746217488879, 1.498747493302027, 1.499583972635738,
1.499970860114983, 1.499907807567145, 1.499394871735262,
1.498432514226959, 1.497021601099038, 1.495163402078079,
1.492859589417777, 1.490112236394023, 1.486923815439098,
1.483297195916649, 1.479235641539457, 1.474742807432315,
1.469822736842662, 1.464479857501934, 1.458718977640905,
1.4525452816626, 1.44596432547669, 1.438982031499539,
1.431604683324436, 1.423838920066784, 1.415691730389341,
1.407170446212898, 1.398282736118043, 1.38903659844396,
1.379440354090461, 1.369502639029735, 1.359232396534563,
1.348638869129968, 1.337731590275575, 1.326520375786132,
1.315015314997945, 1.303226761689157, 1.29116532476204,
1.278841858695708, 1.26626745377781, 1.253453426124026,
1.240411307494323, 1.227152834915152, 1.213689940116914,
1.200034738796209, 1.186199519712527, 1.172196733629194,
1.158038982108526, 1.143739006171271, 1.129309674830555,
1.114763973510631, 1.100114992360884, 1.085375914475572]
```

```
>maxima function fx(t) &= r(t)*cos(t); $'fx(t)=fx(t)
```

```
        fx([0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09,
0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21,
0.2200000000000001, 0.2300000000000001, 0.2400000000000001,
0.2500000000000001, 0.2600000000000001, 0.2700000000000001,
0.2800000000000001, 0.2900000000000001, 0.3000000000000001,
0.3100000000000001, 0.3200000000000001, 0.3300000000000001,
0.3400000000000001, 0.3500000000000001, 0.3600000000000002,
0.3700000000000002, 0.3800000000000002, 0.3900000000000002,
0.4000000000000002, 0.4100000000000002, 0.4200000000000002,
0.4300000000000002, 0.4400000000000002, 0.4500000000000002,
0.4600000000000002, 0.4700000000000003, 0.4800000000000003,
0.4900000000000003, 0.5000000000000002, 0.5100000000000002,
0.5200000000000002, 0.5300000000000002, 0.5400000000000003,
0.5500000000000003, 0.5600000000000003, 0.5700000000000003,
0.5800000000000003, 0.5900000000000003, 0.6000000000000003,
0.6100000000000003, 0.6200000000000003, 0.6300000000000003,
0.6400000000000003, 0.6500000000000004, 0.6600000000000004,
0.6700000000000004, 0.6800000000000004, 0.6900000000000004,
0.7000000000000004, 0.7100000000000004, 0.7200000000000004,
0.7300000000000004, 0.7400000000000004, 0.7500000000000004,
0.7600000000000005, 0.7700000000000005, 0.7800000000000005,
0.7900000000000005, 0.8000000000000005, 0.8100000000000005,
0.8200000000000005, 0.8300000000000005, 0.8400000000000005,
0.8500000000000005, 0.8600000000000005, 0.8700000000000006,
0.8800000000000006, 0.8900000000000006, 0.9000000000000006,
0.9100000000000006, 0.9200000000000006, 0.9300000000000006,
0.9400000000000006, 0.9500000000000006, 0.9600000000000006,
0.9700000000000006, 0.9800000000000006, 0.9900000000000007]) =
                          0.01 a                          0.02 a
[1, 0.9999500004166653 E      , 0.9998000066665778 E      ,
                     0.03 a                          0.04 a
0.9995500337489875 E      , 0.9992001066609779 E      ,
                     0.05 a                          0.06 a
0.9987502603949663 E      , 0.9982005399352042 E      ,
                     0.07 a                          0.08 a
0.9975510002532796 E      , 0.9968017063026194 E      ,
                     0.09 a                           0.1 a
0.9959527330119943 E      , 0.9950041652780258 E     ,
                     0.11 a                          0.12 a
0.9939560979566968 E      , 0.9928086358538663 E      ,
                     0.13 a                          0.14 a
0.9915618937147881 E      , 0.9902159962126372 E      ,
                     0.15 a                          0.16 a
0.9887710779360422 E      , 0.9872272833756269 E      ,
                     0.17 a                          0.18 a
0.9855847669095608 E      , 0.9838436927881214 E      ,
                     0.19 a                           0.2 a
0.9820042351172703 E      , 0.9800665778412416 E     ,
                     0.21 a                          0.2200000000000001 a
0.9780309147241483 E      , 0.9758974493306055 E                   ,
                     0.2300000000000001 a
0.9736663950053748 E                   ,
```

```
                    0.2400000000000001 a
0.9713379748520296 E                    ,
                    0.2500000000000001 a
0.9689124217106447 E                    ,
                    0.2600000000000001 a
0.9663899781345132 E                    ,
                    0.2700000000000001 a
0.9637708963658905 E                    ,
                    0.2800000000000001 a
0.9610554383107709 E                    ,
                    0.2900000000000001 a
0.9582438755126972 E                    ,
                   0.3000000000000001 a
0.955336489125606 E                     ,
                    0.3100000000000001 a
0.9523335698857134 E                    ,
                    0.3200000000000001 a
0.9492354180824408 E                    ,
                    0.3300000000000001 a
0.9460423435283869 E                    ,
                    0.3400000000000001 a
0.9427546655283462 E                    ,
                    0.3500000000000001 a
0.9393727128473789 E                    ,
                    0.3600000000000002 a
0.9358968236779348 E                    ,
                    0.3700000000000002 a
0.9323273456060344 E                    ,
                    0.3800000000000002 a
0.9286646355765101 E                    ,
                   0.3900000000000002 a
0.924909059857313 E                     ,
                   0.4000000000000002 a
0.921060994002885 E                     ,
                   0.4100000000000002 a
0.917120822816605 E                     ,
                    0.4200000000000002 a
0.9130889403123081 E                    ,
                    0.4300000000000002 a
0.9089657496748851 E                    ,
                    0.4400000000000002 a
0.9047516632199634 E                    ,
                    0.4500000000000002 a
0.9004471023526768 E                    ,
                    0.4600000000000002 a
0.8960524975255252 E                    ,
                    0.4700000000000003 a
0.8915682881953289 E                    ,
                   0.4800000000000003 a
0.886994922779284 E                     ,
                    0.4900000000000003 a
0.8823328586101213 E                    ,
                    0.5000000000000002 a
0.8775825618903726 E                    ,
                    0.5100000000000002 a
0.8727445076457512 E                    ,
```

0.5200000000000002 a
0.8678191796776498 E ,
0.5300000000000002 a
0.8628070705147609 E ,
0.5400000000000003 a
0.857708681363824 E ,
0.5500000000000003 a
0.8525245220595056 E ,
0.5600000000000003 a
0.847255111013416 E ,
0.5700000000000003 a
0.8419009751622686 E ,
0.5800000000000003 a
0.8364626499151868 E ,
0.5900000000000003 a
0.8309406791001633 E ,
0.6000000000000003 a
0.8253356149096781 E ,
0.6100000000000003 a
0.8196480178454794 E ,
0.6200000000000003 a
0.8138784566625338 E ,
0.6300000000000003 a
0.8080275083121516 E ,
0.6400000000000003 a
0.8020957578842924 E ,
0.6500000000000004 a
0.7960837985490556 E ,
0.6600000000000004 a
0.7899922314973649 E ,
0.6700000000000004 a
0.783821665880849 E ,
0.6800000000000004 a
0.7775727187509277 E ,
0.6900000000000004 a
0.7712460149971063 E ,
0.7000000000000004 a
0.7648421872844882 E ,
0.7100000000000004 a
0.7583618759905079 E ,
0.7200000000000004 a
0.7518057291408947 E ,
0.7300000000000004 a
0.7451744023448701 E ,
0.7400000000000004 a
0.7384685587295876 E ,
0.7500000000000004 a
0.7316888688738206 E ,
0.7600000000000005 a
0.7248360107409049 E ,
0.7700000000000005 a
0.7179106696109431 E ,
0.7800000000000005 a
0.7109135380122771 E ,
0.7900000000000005 a
0.7038453156522357 E ,

```
                    0.8000000000000005 a
  0.696706709347165 E                   ,
                     0.8100000000000005 a
  0.6894984329517466 E                   ,
                     0.8200000000000005 a
  0.6822212072876132 E                   ,
                     0.8300000000000005 a
  0.6748757600712667 E                   ,
                     0.8400000000000005 a
  0.6674628258413078 E                   ,
                     0.8500000000000005 a
  0.6599831458849817 E                   ,
                     0.8600000000000005 a
  0.6524374681640515 E                   ,
                     0.8700000000000006 a
  0.6448265472400008 E                   ,
                     0.8800000000000006 a
  0.6371511441985798 E                   ,
                     0.8900000000000006 a
  0.6294120265736964 E                   ,
                     0.9000000000000006 a
  0.6216099682706641 E                   ,
                     0.9100000000000006 a
  0.6137457494888111 E                   ,
                     0.9200000000000006 a
  0.6058201566434623 E                   ,
                     0.9300000000000006 a
  0.5978339822872978 E                   ,
                     0.9400000000000006 a
  0.5897880250310977 E                   ,
                    0.9500000000000006 a
  0.581683089463883 E                   ,
                     0.9600000000000006 a
  0.5735199860724561 E                   ,
                     0.9700000000000006 a
  0.5652995311603538 E                   ,
                     0.9800000000000006 a
  0.5570225467662168 E                   ,
                    0.9900000000000007 a
  0.548689860581587 E                   ]
```

```
>maxima function fy(t) &= r(t)*sin(t); $'fy(t)=fy(t)
```

```
          fy([0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09,
  0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21,
  0.2200000000000001, 0.2300000000000001, 0.2400000000000001,
  0.2500000000000001, 0.2600000000000001, 0.2700000000000001,
  0.2800000000000001, 0.2900000000000001, 0.3000000000000001,
  0.3100000000000001, 0.3200000000000001, 0.3300000000000001,
  0.3400000000000001, 0.3500000000000001, 0.3600000000000002,
  0.3700000000000002, 0.3800000000000002, 0.3900000000000002,
  0.4000000000000002, 0.4100000000000002, 0.4200000000000002,
  0.4300000000000002, 0.4400000000000002, 0.4500000000000002,
```

```
0.4600000000000002, 0.4700000000000003, 0.4800000000000003,
0.4900000000000003, 0.5000000000000002, 0.5100000000000002,
0.5200000000000002, 0.5300000000000002, 0.5400000000000003,
0.5500000000000003, 0.5600000000000003, 0.5700000000000003,
0.5800000000000003, 0.5900000000000003, 0.6000000000000003,
0.6100000000000003, 0.6200000000000003, 0.6300000000000003,
0.6400000000000003, 0.6500000000000004, 0.6600000000000004,
0.6700000000000004, 0.6800000000000004, 0.6900000000000004,
0.7000000000000004, 0.7100000000000004, 0.7200000000000004,
0.7300000000000004, 0.7400000000000004, 0.7500000000000004,
0.7600000000000005, 0.7700000000000005, 0.7800000000000005,
0.7900000000000005, 0.8000000000000005, 0.8100000000000005,
0.8200000000000005, 0.8300000000000005, 0.8400000000000005,
0.8500000000000005, 0.8600000000000005, 0.8700000000000006,
0.8800000000000006, 0.8900000000000006, 0.9000000000000006,
0.9100000000000006, 0.9200000000000006, 0.9300000000000006,
0.9400000000000006, 0.9500000000000006, 0.9600000000000006,
0.9700000000000006, 0.9800000000000006, 0.9900000000000007]) =
                          0.01 a                        0.02 a
[0, 0.009999833334166664 E      , 0.01999866669333308 E      ,
                       0.03 a                        0.04 a
0.02999550020249566 E      , 0.03998933418663416 E      ,
                       0.05 a                        0.06 a
0.04997916927067833 E      , 0.0599640064794446 E      ,
                       0.07 a                        0.08 a
0.06994284733753277 E      , 0.0799146939691727 E      ,
                       0.09 a                         0.1 a
0.08987854919801104 E      , 0.09983341664682814 E     ,
                      0.11 a                        0.12 a
0.1097783008371748 E      , 0.1197122072889193 E      ,
                      0.13 a                        0.14 a
0.1296341426196948 E      , 0.1395431146442365 E      ,
                      0.15 a                        0.16 a
0.1494381324735992 E      , 0.159318206614246 E      ,
                     0.17 a                        0.18 a
0.169182349066996 E      , 0.1790295734258242 E      ,
                      0.19 a                         0.2 a
0.1888588949765006 E      , 0.1986693307950612 E     ,
                      0.21 a                         0.2200000000000001 a
0.2084598998460996 E      , 0.2182296230808694 E                    ,
                       0.2300000000000001 a
0.2279775235351885 E                     ,
                       0.2400000000000001 a
0.2377026264271347 E                     ,
                      0.2500000000000001 a
0.247403959254523 E                     ,
                       0.2600000000000001 a
0.2570805518921552 E                     ,
                       0.2700000000000001 a
0.2667314366888312 E                     ,
                       0.2800000000000001 a
0.2763556485641138 E                     ,
                       0.2900000000000001 a
0.2859522251048356 E                     ,
                       0.3000000000000001 a
0.2955202066613397 E                     ,
```

```
                    0.3100000000000001 a
0.3050586364434436 E                    ,
                    0.3200000000000001 a
0.3145665606161179 E                    ,
                    0.3300000000000001 a
0.3240430283948685 E                    ,
                    0.3400000000000001 a
0.3334870921408145 E                    ,
                    0.3500000000000001 a
0.3428978074554515 E                    ,
                    0.3600000000000002 a
0.3522742332750901 E                    ,
                    0.3700000000000002 a
0.3616154319649622 E                    ,
                    0.3800000000000002 a
0.3709204694129828 E                    ,
                    0.3900000000000002 a
0.3801884151231616 E                    ,
                    0.4000000000000002 a
0.3894183423086507 E                    ,
                    0.4100000000000002 a
0.3986093279844231 E                    ,
                    0.4200000000000002 a
0.4077604530595704 E                    ,
                   0.4300000000000002 a
0.416870802429211 E                    ,
                    0.4400000000000002 a
0.4259394650659998 E                    ,
                    0.4500000000000002 a
0.4349655341112304 E                    ,
                  0.4600000000000002 a
0.44394810696552 E                    ,
                    0.4700000000000003 a
0.4528862853790685 E                    ,
                    0.4800000000000003 a
0.4617791755414831 E                    ,
                    0.4900000000000003 a
0.4706258881711582 E                    ,
                    0.5000000000000002 a
0.4794255386042032 E                    ,
                    0.5100000000000002 a
0.4881772468829077 E                    ,
                    0.5200000000000002 a
0.4968801378437369 E                    ,
                    0.5300000000000002 a
0.5055333412048472 E                    ,
                    0.5400000000000003 a
0.5141359916531133 E                    ,
                    0.5500000000000003 a
0.5226872289306594 E                    ,
                    0.5600000000000003 a
0.5311861979208836 E                    ,
                    0.5700000000000003 a
0.5396320487339695 E                    ,
                    0.5800000000000003 a
0.5480239367918738 E                    ,
```

```
                  0.5900000000000003 a
0.556361022912784 E                   ,
                   0.6000000000000003 a
0.5646424733950356 E                    ,
                   0.6100000000000003 a
0.5728674601004815 E                    ,
                   0.6200000000000003 a
0.5810351605373053 E                    ,
                   0.6300000000000003 a
0.5891447579422698 E                    ,
                   0.6400000000000003 a
0.5971954413623923 E                    ,
                   0.6500000000000004 a
0.6051864057360399 E                    ,
                   0.6600000000000004 a
0.6131168519734341 E                    ,
                   0.6700000000000004 a
0.6209859870365599 E                    ,
                   0.6800000000000004 a
0.6287930240184688 E                    ,
                   0.6900000000000004 a
0.6365371822219682 E                    ,
                   0.7000000000000004 a
0.6442176872376913 E                    ,
                  0.7100000000000004 a
0.651833771021537 E                    ,
                   0.7200000000000004 a
0.6593846719714734 E                    ,
                   0.7300000000000004 a
0.6668696350036982 E                    ,
                   0.7400000000000004 a
0.6742879116281454 E                    ,
                   0.7500000000000004 a
0.6816387600233345 E                    ,
                   0.7600000000000005 a
0.6889214451105516 E                    ,
                  0.7700000000000005 a
0.696135238627357 E                    ,
                   0.7800000000000005 a
0.7032794192004105 E                    ,
                   0.7900000000000005 a
0.7103532724176082 E                    ,
                   0.8000000000000005 a
0.7173560908995231 E                    ,
                   0.8100000000000005 a
0.7242871743701429 E                    ,
                   0.8200000000000005 a
0.7311458297268962 E                    ,
                   0.8300000000000005 a
0.7379313711099631 E                    ,
                   0.8400000000000005 a
0.7446431199708596 E                    ,
                  0.8500000000000005 a
0.751280405140293 E                    ,
                   0.8600000000000005 a
0.7578425628952773 E                    ,
```

```
                    0.8700000000000006 a
0.7643289370255054 E                    ,
                    0.8800000000000006 a
0.7707388788989696 E                    ,
                    0.8900000000000006 a
0.7770717475268242 E                    ,
                    0.9000000000000006 a
0.7833269096274837 E                    ,
                    0.9100000000000006 a
0.7895037396899508 E                    ,
                    0.9200000000000006 a
0.7956016200363664 E                    ,
                    0.9300000000000006 a
0.8016199408837775 E                    ,
                    0.9400000000000006 a
0.8075581004051147 E                    ,
                    0.9500000000000006 a
0.8134155047893741 E                    ,
                    0.9600000000000006 a
0.8191915683009986 E                    ,
                    0.9700000000000006 a
0.8248857133384504 E                    ,
                    0.9800000000000006 a
0.8304973704919708 E                    ,
                    0.9900000000000007 a
0.8360259786005209 E                    ]
```

```
>maxima function ds(t) &= trigreduce(radcan(sqrt(diff(fx(t),t)^2+diff(fy(t),t)^2))); $'ds(
```

```
        ds([0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09,
0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21,
0.2200000000000001, 0.2300000000000001, 0.2400000000000001,
0.2500000000000001, 0.2600000000000001, 0.2700000000000001,
0.2800000000000001, 0.2900000000000001, 0.3000000000000001,
0.3100000000000001, 0.3200000000000001, 0.3300000000000001,
0.3400000000000001, 0.3500000000000001, 0.3600000000000002,
0.3700000000000002, 0.3800000000000002, 0.3900000000000002,
0.4000000000000002, 0.4100000000000002, 0.4200000000000002,
0.4300000000000002, 0.4400000000000002, 0.4500000000000002,
0.4600000000000002, 0.4700000000000003, 0.4800000000000003,
0.4900000000000003, 0.5000000000000002, 0.5100000000000002,
0.5200000000000002, 0.5300000000000002, 0.5400000000000003,
0.5500000000000003, 0.5600000000000003, 0.5700000000000003,
0.5800000000000003, 0.5900000000000003, 0.6000000000000003,
0.6100000000000003, 0.6200000000000003, 0.6300000000000003,
0.6400000000000003, 0.6500000000000004, 0.6600000000000004,
0.6700000000000004, 0.6800000000000004, 0.6900000000000004,
0.7000000000000004, 0.7100000000000004, 0.7200000000000004,
0.7300000000000004, 0.7400000000000004, 0.7500000000000004,
0.7600000000000005, 0.7700000000000005, 0.7800000000000005,
0.7900000000000005, 0.8000000000000005, 0.8100000000000005,
0.8200000000000005, 0.8300000000000005, 0.8400000000000005,
0.8500000000000005, 0.8600000000000005, 0.8700000000000006,
```

```
0.8800000000000006, 0.8900000000000006, 0.9000000000000006,
0.9100000000000006, 0.9200000000000006, 0.9300000000000006,
0.9400000000000006, 0.9500000000000006, 0.9600000000000006,
0.9700000000000006, 0.9800000000000006, 0.9900000000000007]) =
ds([0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1,
0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21,
0.2200000000000001, 0.2300000000000001, 0.2400000000000001,
0.2500000000000001, 0.2600000000000001, 0.2700000000000001,
0.2800000000000001, 0.2900000000000001, 0.3000000000000001,
0.3100000000000001, 0.3200000000000001, 0.3300000000000001,
0.3400000000000001, 0.3500000000000001, 0.3600000000000002,
0.3700000000000002, 0.3800000000000002, 0.3900000000000002,
0.4000000000000002, 0.4100000000000002, 0.4200000000000002,
0.4300000000000002, 0.4400000000000002, 0.4500000000000002,
0.4600000000000002, 0.4700000000000003, 0.4800000000000003,
0.4900000000000003, 0.5000000000000002, 0.5100000000000002,
0.5200000000000002, 0.5300000000000002, 0.5400000000000003,
0.5500000000000003, 0.5600000000000003, 0.5700000000000003,
0.5800000000000003, 0.5900000000000003, 0.6000000000000003,
0.6100000000000003, 0.6200000000000003, 0.6300000000000003,
0.6400000000000003, 0.6500000000000004, 0.6600000000000004,
0.6700000000000004, 0.6800000000000004, 0.6900000000000004,
0.7000000000000004, 0.7100000000000004, 0.7200000000000004,
0.7300000000000004, 0.7400000000000004, 0.7500000000000004,
0.7600000000000005, 0.7700000000000005, 0.7800000000000005,
0.7900000000000005, 0.8000000000000005, 0.8100000000000005,
0.8200000000000005, 0.8300000000000005, 0.8400000000000005,
0.8500000000000005, 0.8600000000000005, 0.8700000000000006,
0.8800000000000006, 0.8900000000000006, 0.9000000000000006,
0.9100000000000006, 0.9200000000000006, 0.9300000000000006,
0.9400000000000006, 0.9500000000000006, 0.9600000000000006,
0.9700000000000006, 0.9800000000000006, 0.9900000000000007])
```

```
>$integrate(ds(x),x,0,2*pi) //panjang (keliling) kurva
```

$$\int_0^{2\,\pi} ds\,(x)\;\,dx$$

Maxima gagal melakukan perhitungan eksak integral tersebut.
Berikut kita hitung integralnya secara umerik dengan perintah EMT.

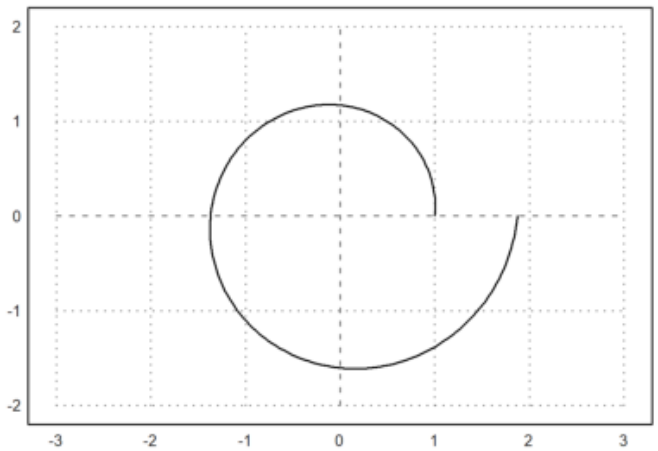
```
>maxima integrate("ds(x)",0,2*pi)
```

```
integrate: variable must not be a number; found: 0
 -- an error. To debug this try: debugmode(true);
```

Spiral Logaritmik

$$x = e^{ax}\cos x,\; y = e^{ax}\sin x.$$

```
>a=0.1; plot2d("exp(a*x)*cos(x)","exp(a*x)*sin(x)",r=2,xmin=0,xmax=2*pi):
```




```
>&kill(a) // hapus expresi a
```

done

```
>maxima function fx(t) &= exp(a*t)*cos(t); $'fx(t)=fx(t)
```

```
        fx([0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09,
  0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21,
  0.2200000000000001, 0.2300000000000001, 0.2400000000000001,
  0.2500000000000001, 0.2600000000000001, 0.2700000000000001,
  0.2800000000000001, 0.2900000000000001, 0.3000000000000001,
  0.3100000000000001, 0.3200000000000001, 0.3300000000000001,
  0.3400000000000001, 0.3500000000000001, 0.3600000000000002,
  0.3700000000000002, 0.3800000000000002, 0.3900000000000002,
  0.4000000000000002, 0.4100000000000002, 0.4200000000000002,
  0.4300000000000002, 0.4400000000000002, 0.4500000000000002,
  0.4600000000000002, 0.4700000000000003, 0.4800000000000003,
  0.4900000000000003, 0.5000000000000002, 0.5100000000000002,
  0.5200000000000002, 0.5300000000000002, 0.5400000000000003,
  0.5500000000000003, 0.5600000000000003, 0.5700000000000003,
  0.5800000000000003, 0.5900000000000003, 0.6000000000000003,
  0.6100000000000003, 0.6200000000000003, 0.6300000000000003,
  0.6400000000000003, 0.6500000000000004, 0.6600000000000004,
  0.6700000000000004, 0.6800000000000004, 0.6900000000000004,
  0.7000000000000004, 0.7100000000000004, 0.7200000000000004,
  0.7300000000000004, 0.7400000000000004, 0.7500000000000004,
  0.7600000000000005, 0.7700000000000005, 0.7800000000000005,
  0.7900000000000005, 0.8000000000000005, 0.8100000000000005,
  0.8200000000000005, 0.8300000000000005, 0.8400000000000005,
```

```
0.8500000000000005, 0.8600000000000005, 0.8700000000000006,
0.8800000000000006, 0.8900000000000006, 0.9000000000000006,
0.9100000000000006, 0.9200000000000006, 0.9300000000000006,
0.9400000000000006, 0.9500000000000006, 0.9600000000000006,
0.9700000000000006, 0.9800000000000006, 0.9900000000000007]) =
                        0.01 a                      0.02 a
[1, 0.9999500004166653 E      , 0.9998000066665778 E      ,
                     0.03 a                      0.04 a
0.9995500337489875 E      , 0.9992001066609779 E      ,
                     0.05 a                      0.06 a
0.9987502603949663 E      , 0.9982005399352042 E      ,
                     0.07 a                      0.08 a
0.9975510002532796 E      , 0.9968017063026194 E      ,
                     0.09 a                      0.1 a
0.9959527330119943 E      , 0.9950041652780258 E     ,
                     0.11 a                      0.12 a
0.9939560979566968 E      , 0.9928086358538663 E      ,
                     0.13 a                      0.14 a
0.9915618937147881 E      , 0.9902159962126372 E      ,
                     0.15 a                      0.16 a
0.9887710779360422 E      , 0.9872272833756269 E      ,
                     0.17 a                      0.18 a
0.9855847669095608 E      , 0.9838436927881214 E      ,
                     0.19 a                      0.2 a
0.9820042351172703 E      , 0.9800665778412416 E     ,
                     0.21 a                      0.2200000000000001 a
0.9780309147241483 E      , 0.9758974493306055 E                    ,
                     0.2300000000000001 a
0.9736663950053748 E                     ,
                     0.2400000000000001 a
0.9713379748520296 E                     ,
                     0.2500000000000001 a
0.9689124217106447 E                     ,
                     0.2600000000000001 a
0.9663899781345132 E                     ,
                     0.2700000000000001 a
0.9637708963658905 E                     ,
                     0.2800000000000001 a
0.9610554383107709 E                     ,
                     0.2900000000000001 a
0.9582438755126972 E                     ,
                    0.3000000000000001 a
0.955336489125606 E                     ,
                     0.3100000000000001 a
0.9523335698857134 E                     ,
                     0.3200000000000001 a
0.9492354180824408 E                     ,
                     0.3300000000000001 a
0.9460423435283869 E                     ,
                     0.3400000000000001 a
0.9427546655283462 E                     ,
                     0.3500000000000001 a
0.9393727128473789 E                     ,
                     0.3600000000000002 a
0.9358968236779348 E                     ,
                     0.3700000000000002 a
```

```
0.9323273456060344 E                    ,
                    0.3800000000000002 a
0.9286646355765101 E                     ,
                   0.3900000000000002 a
0.924909059857313 E                      ,
                   0.4000000000000002 a
0.921060994002885 E                      ,
                   0.4100000000000002 a
0.917120822816605 E                      ,
                    0.4200000000000002 a
0.9130889403123081 E                     ,
                    0.4300000000000002 a
0.9089657496748851 E                     ,
                    0.4400000000000002 a
0.9047516632199634 E                     ,
                    0.4500000000000002 a
0.9004471023526768 E                     ,
                    0.4600000000000002 a
0.8960524975255252 E                     ,
                    0.4700000000000003 a
0.8915682881953289 E                     ,
                   0.4800000000000003 a
0.886994922779284 E                      ,
                    0.4900000000000003 a
0.8823328586101213 E                     ,
                    0.5000000000000002 a
0.8775825618903726 E                     ,
                    0.5100000000000002 a
0.8727445076457512 E                     ,
                    0.5200000000000002 a
0.8678191796776498 E                     ,
                    0.5300000000000002 a
0.8628070705147609 E                     ,
                   0.5400000000000003 a
0.857708681363824 E                      ,
                    0.5500000000000003 a
0.8525245220595056 E                     ,
                   0.5600000000000003 a
0.847255111013416 E                      ,
                    0.5700000000000003 a
0.8419009751622686 E                     ,
                    0.5800000000000003 a
0.8364626499151868 E                     ,
                    0.5900000000000003 a
0.8309406791001633 E                     ,
                    0.6000000000000003 a
0.8253356149096781 E                     ,
                    0.6100000000000003 a
0.8196480178454794 E                     ,
                    0.6200000000000003 a
0.8138784566625338 E                     ,
                    0.6300000000000003 a
0.8080275083121516 E                     ,
                    0.6400000000000003 a
0.8020957578842924 E                     ,
                    0.6500000000000004 a
```

```
0.7960837985490556 E                    ,
                    0.6600000000000004 a
0.7899922314973649 E                     ,
                   0.6700000000000004 a
0.783821665880849 E                      ,
                    0.6800000000000004 a
0.7775727187509277 E                     ,
                    0.6900000000000004 a
0.7712460149971063 E                     ,
                    0.7000000000000004 a
0.7648421872844882 E                     ,
                    0.7100000000000004 a
0.7583618759905079 E                     ,
                    0.7200000000000004 a
0.7518057291408947 E                     ,
                    0.7300000000000004 a
0.7451744023448701 E                     ,
                    0.7400000000000004 a
0.7384685587295876 E                     ,
                    0.7500000000000004 a
0.7316888688738206 E                     ,
                    0.7600000000000005 a
0.7248360107409049 E                     ,
                    0.7700000000000005 a
0.7179106696109431 E                     ,
                    0.7800000000000005 a
0.7109135380122771 E                     ,
                    0.7900000000000005 a
0.7038453156522357 E                     ,
                   0.8000000000000005 a
0.696706709347165 E                      ,
                    0.8100000000000005 a
0.6894984329517466 E                     ,
                    0.8200000000000005 a
0.6822212072876132 E                     ,
                    0.8300000000000005 a
0.6748757600712667 E                     ,
                    0.8400000000000005 a
0.6674628258413078 E                     ,
                    0.8500000000000005 a
0.6599831458849817 E                     ,
                    0.8600000000000005 a
0.6524374681640515 E                     ,
                    0.8700000000000006 a
0.6448265472400008 E                     ,
                    0.8800000000000006 a
0.6371511441985798 E                     ,
                    0.8900000000000006 a
0.6294120265736964 E                     ,
                    0.9000000000000006 a
0.6216099682706641 E                     ,
                    0.9100000000000006 a
0.6137457494888111 E                     ,
                    0.9200000000000006 a
0.6058201566434623 E                     ,
                    0.9300000000000006 a
```

```
0.5978339822872978 E                    ,
                    0.9400000000000006 a
0.5897880250310977 E                     ,
                   0.9500000000000006 a
0.581683089463883 E                     ,
                    0.9600000000000006 a
0.5735199860724561 E                     ,
                    0.9700000000000006 a
0.5652995311603538 E                     ,
                    0.9800000000000006 a
0.5570225467662168 E                     ,
                   0.9900000000000007 a
0.548689860581587 E                     ]
```

```
>maxima function fy(t) &= exp(a*t)*sin(t); $'fy(t)=fy(t)
```

```
        fy([0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09,
0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21,
0.2200000000000001, 0.2300000000000001, 0.2400000000000001,
0.2500000000000001, 0.2600000000000001, 0.2700000000000001,
0.2800000000000001, 0.2900000000000001, 0.3000000000000001,
0.3100000000000001, 0.3200000000000001, 0.3300000000000001,
0.3400000000000001, 0.3500000000000001, 0.3600000000000002,
0.3700000000000002, 0.3800000000000002, 0.3900000000000002,
0.4000000000000002, 0.4100000000000002, 0.4200000000000002,
0.4300000000000002, 0.4400000000000002, 0.4500000000000002,
0.4600000000000002, 0.4700000000000003, 0.4800000000000003,
0.4900000000000003, 0.5000000000000002, 0.5100000000000002,
0.5200000000000002, 0.5300000000000002, 0.5400000000000003,
0.5500000000000003, 0.5600000000000003, 0.5700000000000003,
0.5800000000000003, 0.5900000000000003, 0.6000000000000003,
0.6100000000000003, 0.6200000000000003, 0.6300000000000003,
0.6400000000000003, 0.6500000000000004, 0.6600000000000004,
0.6700000000000004, 0.6800000000000004, 0.6900000000000004,
0.7000000000000004, 0.7100000000000004, 0.7200000000000004,
0.7300000000000004, 0.7400000000000004, 0.7500000000000004,
0.7600000000000005, 0.7700000000000005, 0.7800000000000005,
0.7900000000000005, 0.8000000000000005, 0.8100000000000005,
0.8200000000000005, 0.8300000000000005, 0.8400000000000005,
0.8500000000000005, 0.8600000000000005, 0.8700000000000006,
0.8800000000000006, 0.8900000000000006, 0.9000000000000006,
0.9100000000000006, 0.9200000000000006, 0.9300000000000006,
0.9400000000000006, 0.9500000000000006, 0.9600000000000006,
0.9700000000000006, 0.9800000000000006, 0.9900000000000007]) =
                          0.01 a                        0.02 a
[0, 0.009999833334166664 E      , 0.01999866669333308 E      ,
                     0.03 a                        0.04 a
0.02999550020249566 E      , 0.03998933418663416 E      ,
                     0.05 a                        0.06 a
0.04997916927067833 E      , 0.0599640064794446 E      ,
                     0.07 a                        0.08 a
0.06994284733753277 E      , 0.0799146939691727 E      ,
                     0.09 a                         0.1 a
```

```
0.08987854919801104 E      , 0.09983341664682814 E     ,
                     0.11 a                      0.12 a
0.1097783008371748 E      , 0.1197122072889193 E      ,
                     0.13 a                      0.14 a
0.1296341426196948 E      , 0.1395431146442365 E      ,
                     0.15 a                     0.16 a
0.1494381324735992 E      , 0.159318206614246 E      ,
                    0.17 a                     0.18 a
0.169182349066996 E      , 0.1790295734258242 E      ,
                     0.19 a                      0.2 a
0.1888588949765006 E      , 0.1986693307950612 E     ,
                     0.21 a                      0.2200000000000001 a
0.2084598998460996 E      , 0.2182296230808694 E                    ,
                     0.2300000000000001 a
0.2279775235351885 E                     ,
                     0.2400000000000001 a
0.2377026264271347 E                     ,
                    0.2500000000000001 a
0.247403959254523 E                     ,
                     0.2600000000000001 a
0.2570805518921552 E                     ,
                     0.2700000000000001 a
0.2667314366888312 E                     ,
                     0.2800000000000001 a
0.2763556485641138 E                     ,
                     0.2900000000000001 a
0.2859522251048356 E                     ,
                     0.3000000000000001 a
0.2955202066613397 E                     ,
                     0.3100000000000001 a
0.3050586364434436 E                     ,
                     0.3200000000000001 a
0.3145665606161179 E                     ,
                     0.3300000000000001 a
0.3240430283948685 E                     ,
                     0.3400000000000001 a
0.3334870921408145 E                     ,
                     0.3500000000000001 a
0.3428978074554515 E                     ,
                     0.3600000000000002 a
0.3522742332750901 E                     ,
                     0.3700000000000002 a
0.3616154319649622 E                     ,
                     0.3800000000000002 a
0.3709204694129828 E                     ,
                     0.3900000000000002 a
0.3801884151231616 E                     ,
                     0.4000000000000002 a
0.3894183423086507 E                     ,
                     0.4100000000000002 a
0.3986093279844231 E                     ,
                     0.4200000000000002 a
0.4077604530595704 E                     ,
                    0.4300000000000002 a
0.416870802429211 E                     ,
                     0.4400000000000002 a
```

```
0.4259394650659998 E                    ,
                    0.4500000000000002 a
0.4349655341112304 E                     ,
                  0.4600000000000002 a
0.44394810696552 E                      ,
                    0.4700000000000003 a
0.4528862853790685 E                     ,
                    0.4800000000000003 a
0.4617791755414831 E                     ,
                    0.4900000000000003 a
0.4706258881711582 E                     ,
                    0.5000000000000002 a
0.4794255386042032 E                     ,
                    0.5100000000000002 a
0.4881772468829077 E                     ,
                    0.5200000000000002 a
0.4968801378437369 E                     ,
                    0.5300000000000002 a
0.5055333412048472 E                     ,
                    0.5400000000000003 a
0.5141359916531133 E                     ,
                    0.5500000000000003 a
0.5226872289306594 E                     ,
                    0.5600000000000003 a
0.5311861979208836 E                     ,
                    0.5700000000000003 a
0.5396320487339695 E                     ,
                    0.5800000000000003 a
0.5480239367918738 E                     ,
                   0.5900000000000003 a
0.556361022912784 E                      ,
                    0.6000000000000003 a
0.5646424733950356 E                     ,
                    0.6100000000000003 a
0.5728674601004815 E                     ,
                    0.6200000000000003 a
0.5810351605373053 E                     ,
                    0.6300000000000003 a
0.5891447579422698 E                     ,
                    0.6400000000000003 a
0.5971954413623923 E                     ,
                    0.6500000000000004 a
0.6051864057360399 E                     ,
                    0.6600000000000004 a
0.6131168519734341 E                     ,
                    0.6700000000000004 a
0.6209859870365599 E                     ,
                    0.6800000000000004 a
0.6287930240184688 E                     ,
                    0.6900000000000004 a
0.6365371822219682 E                     ,
                    0.7000000000000004 a
0.6442176872376913 E                     ,
                   0.7100000000000004 a
0.651833771021537 E                      ,
                    0.7200000000000004 a
```

```
0.6593846719714734 E                    ,
                    0.7300000000000004 a
0.6668696350036982 E                    ,
                    0.7400000000000004 a
0.6742879116281454 E                    ,
                    0.7500000000000004 a
0.6816387600233345 E                    ,
                    0.7600000000000005 a
0.6889214451105516 E                    ,
                   0.7700000000000005 a
0.696135238627357 E                     ,
                    0.7800000000000005 a
0.7032794192004105 E                    ,
                    0.7900000000000005 a
0.7103532724176082 E                    ,
                    0.8000000000000005 a
0.7173560908995231 E                    ,
                    0.8100000000000005 a
0.7242871743701429 E                    ,
                    0.8200000000000005 a
0.7311458297268962 E                    ,
                    0.8300000000000005 a
0.7379313711099631 E                    ,
                    0.8400000000000005 a
0.7446431199708596 E                    ,
                   0.8500000000000005 a
0.751280405140293 E                     ,
                    0.8600000000000005 a
0.7578425628952773 E                    ,
                    0.8700000000000006 a
0.7643289370255054 E                    ,
                    0.8800000000000006 a
0.7707388788989696 E                    ,
                    0.8900000000000006 a
0.7770717475268242 E                    ,
                    0.9000000000000006 a
0.7833269096274837 E                    ,
                    0.9100000000000006 a
0.7895037396899508 E                    ,
                    0.9200000000000006 a
0.7956016200363664 E                    ,
                    0.9300000000000006 a
0.8016199408837775 E                    ,
                    0.9400000000000006 a
0.8075581004051147 E                    ,
                    0.9500000000000006 a
0.8134155047893741 E                    ,
                    0.9600000000000006 a
0.8191915683009986 E                    ,
                    0.9700000000000006 a
0.8248857133384504 E                    ,
                    0.9800000000000006 a
0.8304973704919708 E                    ,
                    0.9900000000000007 a
0.8360259786005209 E                     ]
```

```
>maxima function df(t) &= trigreduce(radcan(sqrt(diff(fx(t),t)^2+diff(fy(t),t)^2))); $'df(
```

```
        df([0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09,
0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19, 0.2, 0.21,
0.2200000000000001, 0.2300000000000001, 0.2400000000000001,
0.2500000000000001, 0.2600000000000001, 0.2700000000000001,
0.2800000000000001, 0.2900000000000001, 0.3000000000000001,
0.3100000000000001, 0.3200000000000001, 0.3300000000000001,
0.3400000000000001, 0.3500000000000001, 0.3600000000000002,
0.3700000000000002, 0.3800000000000002, 0.3900000000000002,
0.4000000000000002, 0.4100000000000002, 0.4200000000000002,
0.4300000000000002, 0.4400000000000002, 0.4500000000000002,
0.4600000000000002, 0.4700000000000003, 0.4800000000000003,
0.4900000000000003, 0.5000000000000002, 0.5100000000000002,
0.5200000000000002, 0.5300000000000002, 0.5400000000000003,
0.5500000000000003, 0.5600000000000003, 0.5700000000000003,
0.5800000000000003, 0.5900000000000003, 0.6000000000000003,
0.6100000000000003, 0.6200000000000003, 0.6300000000000003,
0.6400000000000003, 0.6500000000000004, 0.6600000000000004,
0.6700000000000004, 0.6800000000000004, 0.6900000000000004,
0.7000000000000004, 0.7100000000000004, 0.7200000000000004,
0.7300000000000004, 0.7400000000000004, 0.7500000000000004,
0.7600000000000005, 0.7700000000000005, 0.7800000000000005,
0.7900000000000005, 0.8000000000000005, 0.8100000000000005,
0.8200000000000005, 0.8300000000000005, 0.8400000000000005,
0.8500000000000005, 0.8600000000000005, 0.8700000000000006,
0.8800000000000006, 0.8900000000000006, 0.9000000000000006,
0.9100000000000006, 0.9200000000000006, 0.9300000000000006,
0.9400000000000006, 0.9500000000000006, 0.9600000000000006,
0.9700000000000006, 0.9800000000000006, 0.9900000000000007]) =
[0, - 0.2799760005866601, - 0.5598080187724881, - 0.8393521425455596,
- 1.118464600638492, - 1.397001832817543, - 1.67482056007196,
- 1.951777854670376, - 2.227731210051152, - 2.502538610513614,
- 2.776058600677231, - 3.048150354675898, - 3.318673745054589,
- 3.587489411335808, - 3.854458828223397, - 4.119444373411438,
- 4.382309394966176, - 4.642918278249049, - 4.901136512349193,
- 5.156830755993941, - 5.409868902906116, - 5.660120146577182,
- 5.907455044425564, - 6.151745581309722, - 6.392865232365909,
- 6.630689025140811, - 6.865093600989598, - 7.095957275710289,
- 7.323160099385628, - 7.546583915404101, - 7.76611241863204,
- 7.981631212709207, - 8.193027866440602, - 8.40019196925773,
- 8.603015185722887, - 8.80139130905058, - 8.99521631362057,
- 9.184388406457535, - 9.368808077652794, - 9.548378149704078,
- 9.72300382574974, - 9.89259273667444, - 10.05705498706371,
- 10.2163031999855, - 10.37025256057714, - 10.518820858417,
- 10.66192852866037, - 10.79949869191984, - 10.9314571928711,
- 11.05773263756544, - 11.17825642943104, - 11.29296280394572,
- 11.40178886196424, - 11.50467460168416, - 11.60156294923465,
- 11.6923997878735, - 11.7771339857779, - 11.85571742241566,
- 11.92810501348376, - 11.99425473440209, - 12.05412764235074,
- 12.10768789683997, - 12.15490277880263, - 12.19574270819948,
- 12.23018126012851, - 12.25819517943024, - 12.27976439378145,
- 12.2948720252705, - 12.30350440044846, - 12.30565105885034,
- 12.30130475998222, - 12.29046148876983, - 12.27312045946609,
```

```
- 12.24928411801434, - 12.21895814286603, - 12.18215144425176,
- 12.13887616190507, - 12.08914766123974, - 12.03298452798139,
- 11.97040856125544, - 11.90144476513393, - 11.82612133864437,
- 11.74446966424476, - 11.65652429476939, - 11.56232293885094,
- 11.46190644482483, - 11.35531878312292, - 11.24260702716385,
- 11.12382133274848, - 10.99901491596926, - 10.86824402964322,
- 10.73156793827891, - 10.58904889158832, - 10.44075209655548,
- 10.28674568807416, - 10.12710069816765, - 9.961891023804439,
- 9.791193393324098, - 9.615087331488441, - 9.433655123173638]
```

```
>maxima &=integrate(df(t),t,0,2*%pi); $S // panjang kurva (spiral)
```

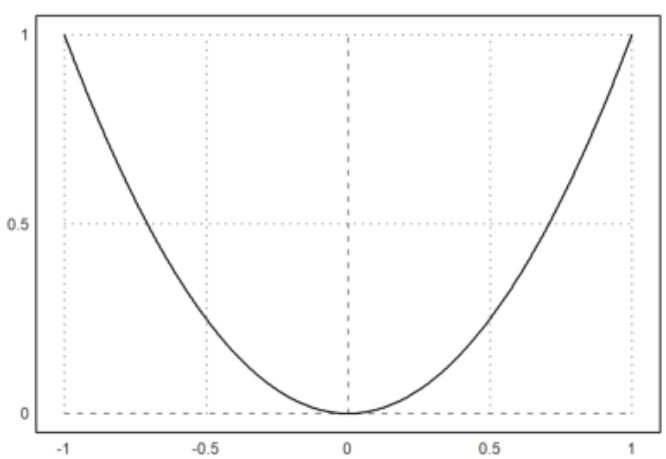
S

```
>maxima (a=0.1) // Panjang kurva untuk a=0.1
```

a = 0.1

Soal:
Tunjukkan bahwa keliling lingkaran dengan jari-jari r adalah K=2.pi.r.
Berikut adalah contoh menghitung panjang parabola.

```
>plot2d("x^2",xmin=-1,xmax=1):
```



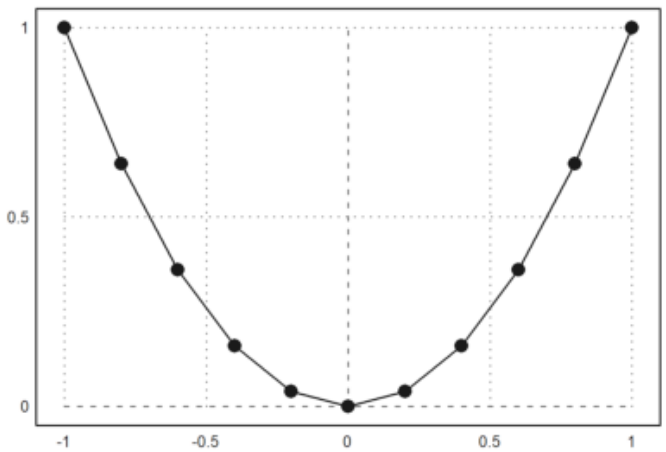

```
>$showev('integrate(sqrt(1+diff(x^2,x)^2),x,-1,1))
```

$$\int_{-1}^{1} \sqrt{4\,x^2+1}\,dx = \frac{\operatorname{asinh} 2+2\sqrt{5}}{2}$$

```
>$float(%)
```

$$\int_{-1.0}^{1.0} \sqrt{4.0\,x^2 + 1.0}\,dx = 2.957885715089195$$

```
>x=-1:0.2:1; y=x^2; plot2d(x,y);  ...
>  plot2d(x,y,points=1,style="o#",add=1):
```



Panjang tersebut dapat dihampiri dengan menggunakan jumlah panjang ruas-ruas garis yang menghubungkan titik-titik pada parabola tersebut.

```
>i=1:cols(x)-1; sum(sqrt((x[i+1]-x[i])^2+(y[i+1]-y[i])^2))
```

```
  2.95191957027
```

Hasilnya mendekati panjang yang dihitung secara eksak. Untuk mendapatkan hampiran yang cukup akurat, jarak antar titik dapat diperkecil, misalnya 0.1, 0.05, 0.01, dan seterusnya. Cobalah Anda ulangi perhitungannya dengan nilai-nilai tersebut.

Koordinat Kartesius

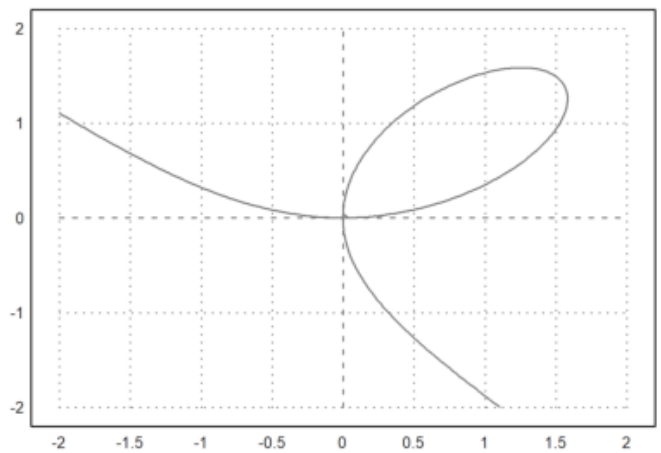
Berikut diberikan contoh perhitungan panjang kurva menggunakan koordinat Kartesius. Kita akan hitung panjang kurva dengan persamaan implisit:

$$x^3 + y^3 - 3xy = 0.$$

```
>z &= x^3+y^3-3*x*y; $z
```

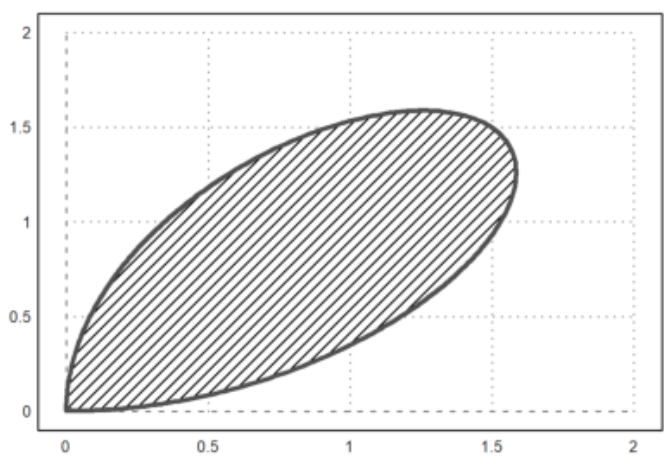
$$y^3 - 3\,x\,y + x^3$$

```
>plot2d(z,r=2,level=0,n=100):
```



Kita tertarik pada kurva di kuadran pertama.

```
>plot2d(z,a=0,b=2,c=0,d=2,level=[-10;0],n=100,contourwidth=3,style="/"):
```



Kita selesaikan persamaannya untuk x.

```
>$z with y=l*x, sol &= solve(%,x); $sol
```

$$l^3\,x^3+x^3-3\,l\,x^2$$

$$\left[x=\frac{3\,l}{l^3+1},x=0\right]$$

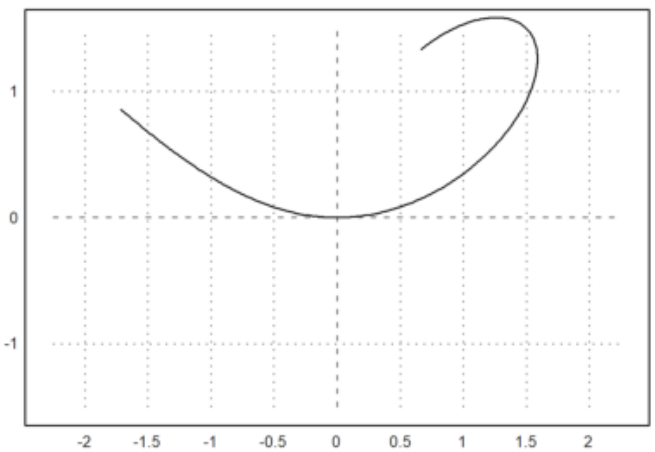
Kita gunakan solusi tersebut untuk mendefinisikan fungsi dengan Maxima.

```
>function f(l) &= rhs(sol[1]); $'f(l)=f(l)
```

$$f(l) = \frac{3\,l}{l^3+1}$$

Fungsi tersebut juga dapat digunaka untuk menggambar kurvanya. Ingat, bahwa fungsi tersebut adalah nilai x dan nilai y=l*x, yakni x=f(l) dan y=l*f(l).

```
>plot2d(&f(x),&x*f(x),xmin=-0.5,xmax=2,a=0,b=2,c=0,d=2,r=1.5):
```



Elemen panjang kurva adalah:

$$ds = \sqrt{f'(l)^2 + (lf'(l) + f(l))^2}.$$

```
>function ds(l) &= ratsimp(sqrt(diff(f(l),l)^2+diff(l*f(l),l)^2)); $'ds(l)=ds(l)
```

$$ds\,(l) = \frac{\sqrt{9\,l^8+36\,l^6-36\,l^5-36\,l^3+36\,l^2+9}}{\sqrt{l^{12}+4\,l^9+6\,l^6+4\,l^3+1}}$$

```
>$integrate(ds(l),l,0,1)
```

$$\int_0^1 \frac{\sqrt{9\,l^8+36\,l^6-36\,l^5-36\,l^3+36\,l^2+9}}{\sqrt{l^{12}+4\,l^9+6\,l^6+4\,l^3+1}}\,dl$$

Integral tersebut tidak dapat dihitung secara eksak menggunakan Maxima. Kita hitung integral etrsebut secara numerik dengan Euler. Karena kurva simetris, kita hitung untuk nilai variabel integrasi dari 0 sampai 1, kemudian hasilnya dikalikan 2.

```
>2*integrate("ds(x)",0,1)
```

```
  4.91748872168
```

```
>2*romberg(&ds(x),0,1)// perintah Euler lain untuk menghitung nilai hampiran integral yang
```

```
  4.91748872168
```

Perhitungan di datas dapat dilakukan untuk sebarang fungsi x dan y dengan mendefinisikan fungsi EMT, misalnya kita beri nama panjangkurva. Fungsi ini selalu memanggil Maxima untuk menurunkan fungsi yang diberikan.

```
>function panjangkurva(fx,fy,a,b) ...
```

```
  ds=mxm("sqrt(diff(@fx,x)^2+diff(@fy,x)^2)");
  return romberg(ds,a,b);
  endfunction
```

```
>panjangkurva("x","x^2",-1,1) // cek untuk menghitung panjang kurva parabola sebelumnya
```

```
  2.95788571509
```

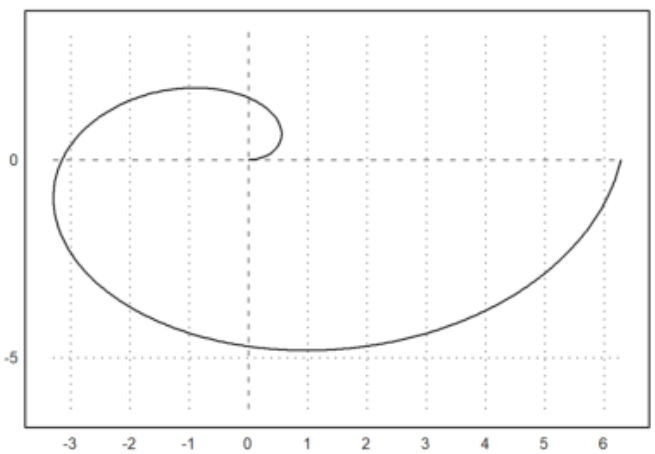
Bandingkan dengan nilai eksak di atas.

```
>2*panjangkurva(mxm("f(x)"),mxm("x*f(x)"),0,1) // cek contoh terakhir, bandingkan hasilnya
```

```
  4.91748872168
```

Kita hitung panjang spiral Archimides berikut ini dengan fungsi tersebut.

```
>plot2d("x*cos(x)","x*sin(x)",xmin=0,xmax=2*pi,square=1):
```



```
>panjangkurva("x*cos(x)","x*sin(x)",0,2*pi)
```

```
21.2562941482
```

Berikut kita definisikan fungsi yang sama namun dengan Maxima, untuk perhitungan eksak.

```
>&kill(ds,x,fx,fy)
```

```
done
```

```
>function ds(fx,fy) &&= sqrt(diff(fx,x)^2+diff(fy,x)^2)
```

```
                    2                2
sqrt(diff (fy, x) + diff (fx, x))
```

```
>sol &= ds(x*cos(x),x*sin(x)); $sol // Kita gunakan untuk menghitung panjang kurva terakhi
```

$$\sqrt{(\cos x - x\,\sin x)^2 + (\sin x + x\,\cos x)^2}$$

```
>$sol | trigreduce | expand, $integrate(%,x,0,2*pi), %()
```

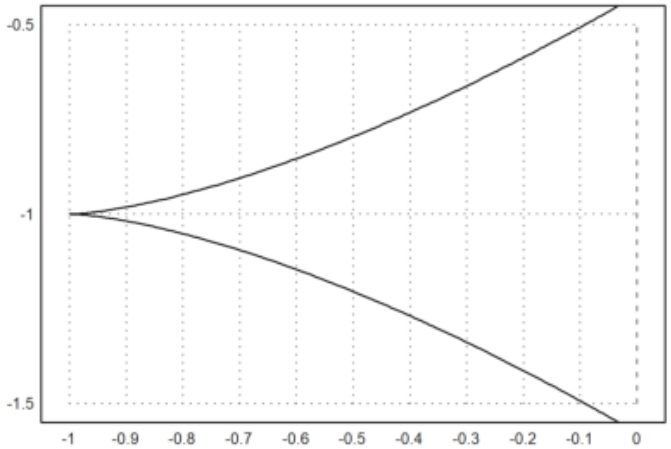
$$\sqrt{x^2+1}$$

$$\frac{\operatorname{asinh}\,(2\,\pi) + 2\,\pi\,\sqrt{4\,\pi^2+1}}{2}$$

```
21.2562941482
```

Hasilnya sama dengan perhitungan menggunakan fungsi EMT.
Berikut adalah contoh lain penggunaan fungsi Maxima tersebut.

```
>plot2d("3*x^2-1","3*x^3-1",xmin=-1/sqrt(3),xmax=1/sqrt(3),square=1):
```




```
>sol &= radcan(ds(3*x^2-1,3*x^3-1)); $sol
```

$$3\,x\,\sqrt{9\,x^2+4}$$

```
>$showev('integrate(sol,x,0,1/sqrt(3))), $2*float(%) // panjang kurva di atas
```

$$3\int_0^{\frac{1}{\sqrt{3}}} x\,\sqrt{9\,x^2+4}\,dx = 3\left(\frac{7^{\frac{3}{2}}}{27}-\frac{8}{27}\right)$$

$$6.0\int_{0.0}^{0.5773502691896258} x\,\sqrt{9.0\,x^2+4.0}\,dx = 2.337835372767141$$

Sikloid

Berikut kita akan menghitung panjang kurva lintasan (sikloid) suatu titik pada lingkaran yang berputar ke kanan pada permukaan datar. Misalkan jari-jari lingkaran tersebut adalah r. Posisi titik pusat lingkaran pada saat t adalah:

$$(rt, r).$$

Misalkan posisi titik pada lingkaran tersebut mula-mula (0,0) dan posisinya pada saat t adalah:

$$(r(t-\sin(t)), r(1-\cos(t))).$$

Berikut kita plot lintasan tersebut dan beberapa posisi lingkaran ketika t=0, t=pi/2, t=r*pi.

```
>x &= r*(t-sin(t))
```

```
          [0, 1.66665833335744e-7 r, 1.33330666692022e-6 r,
  4.499797504338432e-6 r, 1.066581336583994e-5 r,
  2.083072932167196e-5 r, 3.599352055540239e-5 r,
```

```
5.71526624672386e-5 r, 8.530603082730626e-5 r,
1.214508019889565e-4 r, 1.665833531718508e-4 r,
2.216991628251896e-4 r, 2.877927110806339e-4 r,
3.658573803051457e-4 r, 4.568853557635201e-4 r,
5.618675264007778e-4 r, 6.817933857540259e-4 r,
8.176509330039827e-4 r, 9.704265741758145e-4 r,
0.001141105023499428 r, 0.001330669204938795 r,
0.001540100153900437 r, 0.001770376919130678 r,
0.002022476464811601 r, 0.002297373572865413 r,
0.002596040745477063 r, 0.002919448107844891 r,
0.003268563311168871 r, 0.003644351435886262 r,
0.004047774895164447 r, 0.004479793338660443 r, 0.0049413635565565 r,
0.005433439383882244 r, 0.005956971605131645 r,
0.006512907859185624 r, 0.007102192544548636 r,
0.007725766724910044 r, 0.00838456803503801 r,
0.009079530587017326 r, 0.009811584876838586 r, 0.0105816576913495 r,
0.01139067201557714 r, 0.01223954694042984 r, 0.01312919757078923 r,
0.01406053493400045 r, 0.01503446588876983 r, 0.01605189303448024 r,
0.01711371462093175 r, 0.01822082445851714 r, 0.01937411182884202 r,
0.02057446139579705 r, 0.02182275311709253 r, 0.02311986215626333 r,
0.02446665879515308 r, 0.02586400834688696 r, 0.02731277106934082 r,
0.02881380207911666 r, 0.03036795126603076 r, 0.03197606320812652 r,
0.0336389770872163 r, 0.03535752660496472 r, 0.03713253989951881 r,
0.03896483946269502 r, 0.0408552420577305 r, 0.04280455863760801 r,
0.04481359426396048 r, 0.04688314802656623 r, 0.04901401296344043 r,
0.05120697598153157 r, 0.05346281777803219 r, 0.05578231276230905 r,
0.05816622897846346 r, 0.06061532802852698 r, 0.0631303649963022 r,
0.06571208837185505 r, 0.06836123997666599 r, 0.07107855488944881 r,
0.07386476137264342 r, 0.07672058079958999 r, 0.07964672758239233 r,
0.08264390910047736 r, 0.0857128256298576 r, 0.08885417027310427 r,
0.09206862889003742 r, 0.09535688002914089 r, 0.0987195948597075 r,
0.1021574371047232 r, 0.1056710629744951 r, 0.1092611211010309 r,
0.1129282524731764 r, 0.1166730903725168 r, 0.1204962603100498 r,
0.1243983799636342 r, 0.1283800591162231 r, 0.1324418995948859 r,
0.1365844952106265 r, 0.140808431699002 r, 0.1451142866615502 r,
0.1495026295080298 r, 0.1539740213994798 r]
```

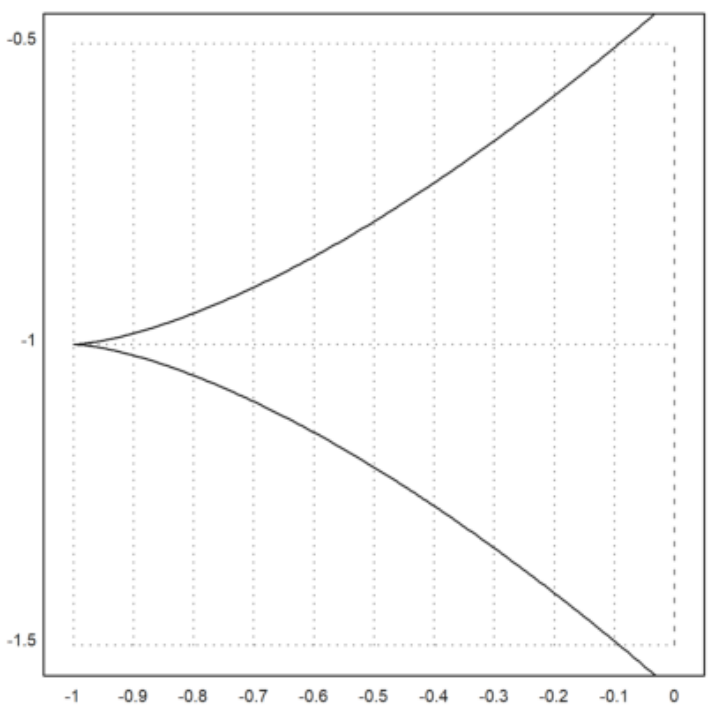
```
>y &= r*(1-cos(t))
```

```
        [0, 4.999958333473664e-5 r, 1.999933334222437e-4 r,
4.499662510124569e-4 r, 7.998933390220841e-4 r,
0.001249739605033717 r, 0.00179946006479581 r,
0.002448999746720415 r, 0.003198293697380561 r,
0.004047266988005727 r, 0.004995834721974179 r,
0.006043902043303184 r, 0.00719136414613375 r, 0.00843810628521191 r,
0.009784003787362772 r, 0.01122892206395776 r, 0.01277271662437307 r,
0.01441523309043924 r, 0.01615630721187855 r, 0.01799576488272969 r,
0.01993342215875837 r, 0.02196908527585173 r, 0.02410255066939448 r,
0.02633360499462523 r, 0.02866202514797045 r, 0.03108757828935527 r,
0.03361002186548678 r, 0.03622910363410947 r, 0.03894456168922911 r,
0.04175612448730281 r, 0.04466351087439402 r, 0.04766643011428662 r,
0.05076458191755917 r, 0.0539576564716131 r, 0.05724533447165381 r,
0.06062728715262111 r, 0.06410317632206519 r, 0.06767265439396564 r,
```

```
0.07133536442348987 r, 0.07509094014268702 r, 0.07893900599711501 r,
0.08287917718339499 r, 0.08691105968769186 r, 0.09103425032511492 r,
0.09524833678003664 r, 0.09955289764732322 r, 0.1039475024744748 r,
0.1084317118046711 r, 0.113005077220716 r, 0.1176671413898787 r,
0.1224174381096274 r, 0.1272554923542488 r, 0.1321808203223502 r,
0.1371929294852391 r, 0.1422913186361759 r, 0.1474754779404944 r,
0.152744888986584 r, 0.1580990248377314 r, 0.1635373500848132 r,
0.1690593208998367 r, 0.1746643850903219 r, 0.1803519821545206 r,
0.1861215433374662 r, 0.1919724916878484 r, 0.1979042421157076 r,
0.2039162014509444 r, 0.2100077685026351 r, 0.216178334119151 r,
0.2224272812490723 r, 0.2287539850028937 r, 0.2351578127155118 r,
0.2416381240094921 r, 0.2481942708591053 r, 0.2548255976551299 r,
0.2615314412704124 r, 0.2683111311261794 r, 0.2751639892590951 r,
0.2820893303890569 r, 0.2890864619877229 r, 0.2961546843477643 r,
0.3032932906528349 r, 0.3105015670482534 r, 0.3177787927123868 r,
0.3251242399287333 r, 0.3325371741586922 r, 0.3400168541150183 r,
0.3475625318359485 r, 0.3551734527599992 r, 0.3628488558014202 r,
0.3705879734263036 r, 0.3783900317293359 r, 0.3862542505111889 r,
0.3941798433565377 r, 0.4021660177127022 r, 0.4102119749689023 r,
0.418316910536117 r, 0.4264800139275439 r, 0.4347004688396462 r,
0.4429774532337832 r, 0.451310139418413 r]
```

Berikut kita gambar sikloid untuk r=1.

```
>ex &= x-sin(x); ey &= 1-cos(x); aspect(1):
```



```
>plot2d(ex,ey,xmin=0,xmax=4pi,square=1): ...
>  plot2d("2+cos(x)","1+sin(x)",xmin=0,xmax=2pi,>add,color=blue): ...
>  plot2d([2,ex(2)],[1,ey(2)],color=red,>add): ...
>  plot2d(ex(2),ey(2),>points,>add,color=red): ...
>  plot2d("2pi+cos(x)","1+sin(x)",xmin=0,xmax=2pi,>add,color=blue): ...
>  plot2d([2pi,ex(2pi)],[1,ey(2pi)],color=red,>add):  ...
>  plot2d(ex(2pi),ey(2pi),>points,>add,color=red):
```

```
  Error : [0,1.66665833335744e-7*r-sin(1.66665833335744e-7*r),1.33330666692022e-6*r-sin(1.

  Error generated by error() command

  adaptiveeval:
      error(f$|" does not produce a real or column vector");
  Try "trace errors" to inspect local variables after errors.
  plot2d:
      dw/n,dw/n^2,dw/n;args());
```

Berikut dihitung panjang lintasan untuk 1 putaran penuh. (Jangan salah menduga bahwa panjang lintasan 1 putaran penuh sama dengan keliling lingkaran!)

```
>maxima ds &= radcan(sqrt(diff(ex,x)^2+diff(ey,x)^2)); $ds=trigsimp(ds) // elemen panjang
```

$$ds = ds$$

```
>maxima ds &= trigsimp(ds); $ds
```

$$ds$$

```
>$showev('integrate(ds,x,0,2*pi)) // hitung panjang sikloid satu putaran penuh
```

```
  Maxima said:
  defint: variable of integration must be a simple or subscripted variable.
  defint: found errexp1
  #0: showev(f='integrate(ds,[0,1.66665833335744e-7*r,1.33330666692022e-6*r,4.499797504338
   -- an error. To debug this try: debugmode(true);

  Error in:
  $showev('integrate(ds,x,0,2*pi)) // hitung panjang sikloid sat ...
                                   ^
```

```
>integrate(mxm("ds"),0,2*pi) // hitung secara numerik
```

```
  Illegal function result in map.
  %evalexpression:
      if maps then return %mapexpression1(x,f$;args());
  gauss:
      if maps then y=%evalexpression(f$,a+h-(h*xn)',maps;args());
  adaptivegauss:
      t1=gauss(f$,c,c+h;args(),=maps);
  Try "trace errors" to inspect local variables after errors.
  integrate:
      return adaptivegauss(f$,a,b,eps*1000;args(),=maps);
```

```
>romberg(mxm("ds"),0,2*pi) // cara lain hitung secara numerik
```

```
  Wrong argument!

  Cannot combine a symbolic expression here.
  Did you want to create a symbolic expression?
  Then start with &.

  Try "trace errors" to inspect local variables after errors.
  romberg:
      if cols(y)==1 then return y*(b-a); endif;
  Error in:
  romberg(mxm("ds"),0,2*pi) // cara lain hitung secara numerik ...
                            ^
```

Perhatikan, seperti terlihat pada gambar, panjang sikloid lebih besar daripada keliling lingkarannya, yakni:

$$2\pi.$$

Kurvatur (Kelengkungan) Kurva

image: Osculating.png

Aslinya, kelengkungan kurva diferensiabel (yakni, kurva mulus yang tidak lancip) di titik P didefinisikan melalui lingkaran oskulasi (yaitu, lingkaran yang melalui titik P dan terbaik memperkirakan, paling banyak menyinggung kurva di sekitar P). Pusat dan radius kelengkungan kurva di P adalah pusat dan radius lingkaran oskulasi. Kelengkungan adalah kebalikan dari radius kelengkungan:

$$\kappa = \frac{1}{R}$$

dengan R adalah radius kelengkungan. (Setiap lingkaran memiliki kelengkungan ini pada setiap titiknya, dapat diartikan, setiap lingkaran berputar 2pi sejauh 2piR.)
Definisi ini sulit dimanipulasi dan dinyatakan ke dalam rumus untuk kurva umum. Oleh karena itu digunakan definisi lain yang ekivalen.

Definisi Kurvatur dengan Fungsi Parametrik Panjang Kurva

Setiap kurva diferensiabel dapat dinyatakan dengan persamaan parametrik terhadap panjang kurva s:

$$\gamma(s) = (x(s),\ y(s)),$$

dengan x dan y adalah fungsi riil yang diferensiabel, yang memenuhi:

$$\|\gamma'(s)\| = \sqrt{x'(s)^2 + y'(s)^2} = 1.$$

Ini berarti bahwa vektor singgung

$$\mathbf{T}(s) = (x'(s),\ y'(s))$$

memiliki norm 1 dan merupakan vektor singgung satuan.

Apabila kurvanya memiliki turunan kedua, artinya turunan kedua x dan y ada, maka T'(s) ada. Vektor ini merupakan normal kurva yang arahnya menuju pusat kurvatur, norm-nya merupakan nilai kurvatur (kelengkungan):

$$\begin{aligned}\mathbf{T}(s) &= \gamma'(s),\\ \mathbf{T}^2(s) &= 1 \text{ (konstanta)} \Rightarrow \mathbf{T}'(s)\cdot\mathbf{T}(s) = 0\\ \kappa(s) &= \|\mathbf{T}'(s)\| = \|\gamma''(s)\| = \sqrt{x''(s)^2 + y''(s)^2}.\end{aligned}$$

Nilai

$$R(s) = \frac{1}{\kappa(s)}$$

disebut jari-jari (radius) kelengkungan kurva.

Bilangan riil

$$k(s) = \pm\kappa(s)$$

disebut nilai kelengkungan bertanda.

Contoh:
Akan ditentukan kurvatur lingkaran

$$x = r\cos t,\ y = r\sin t.$$

```
>maxima fx &= r*cos(t); fy &=r*sin(t);
>maxima &assume(t>0,r>0); s &=integrate(sqrt(diff(fx,t)^2+diff(fy,t)^2),t,0,t); s // eleme
```

s

```
>maxima &kill(s); fx &= r*cos(s/r); fy &=r*sin(s/r); // definisi ulang persamaan parametri
>maxima k &= trigsimp(sqrt(diff(fx,s,2)^2+diff(fy,s,2)^2)); $k // nilai kurvatur lingkaran
```

k

Untuk representasi parametrik umum, misalkan

$$x = x(t),\ y = y(t)$$

merupakan persamaan parametrik untuk kurva bidang yang terdiferensialkan dua kali. Kurvatur untuk kurva tersebut didefinisikan sebagai

$$\kappa = \frac{d\phi}{ds} = \frac{\frac{d\phi}{dt}}{\frac{ds}{dt}} \quad (\phi \text{ adalah sudut kemiringan garis singgung dan } s \text{ adalah panjang kurva})$$
$$= \frac{\frac{d\phi}{dt}}{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}} = \frac{\frac{d\phi}{dt}}{\sqrt{x'(t)^2 + y'(t)^2}}.$$

Selanjutnya, pembilang pada persamaan di atas dapat dicari sebagai berikut.

$$\sec^2\phi \frac{d\phi}{dt} = \frac{d}{dt}(\tan\phi) = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right) = \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) = \frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2}.$$

$$\begin{aligned}
\frac{d\phi}{dt} &= \frac{1}{\sec^2\phi}\frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2} \\
&= \frac{1}{1+\tan^2\phi}\frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2} \\
&= \frac{1}{1+\left(\frac{y'(t)}{x'(t)}\right)^2}\frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2} \\
&= \frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2 + y'(t)^2}.
\end{aligned}$$

Jadi, rumus kurvatur untuk kurva parametrik

$$x = x(t),\ y = y(t)$$

adalah

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}.$$

Jika kurvanya dinyatakan dengan persamaan parametrik pada koordinat kutub

$$x = r(\theta)\cos\theta,\ y = r(\theta)\sin\theta,$$

maka rumus kurvaturnya adalah

$$\kappa(\theta) = \frac{r(\theta)^2 + 2r'(\theta)^2 - r(\theta)r''(\theta)}{(r'(\theta)^2 + r'(\theta)^2)^{3/2}}.$$

(Silakan Anda turunkan rumus tersebut!)
Contoh:
Lingkaran dengan pusat (0,0) dan jari-jari r dapat dinyatakan dengan persamaan parametrik

$$x = r\cos t,\ y = r\sin t.$$

Nilai kelengkungan lingkaran tersebut adalah

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}} = \frac{r^2}{r^3} = \frac{1}{r}.$$

Hasil cocok dengan definisi kurvatur suatu kelengkungan.

Kurva

$$y = f(x)$$

dapat dinyatakan ke dalam persamaan parametrik

$$x = t,\ y = f(t),\ \text{ dengan } x'(t) = 1,\ x''(t) = 0,$$

sehingga kurvaturnya adalah

$$\kappa(t) = \frac{y''(t)}{(1 + y'(t)^2)^{3/2}}.$$

Contoh:
Akan ditentukan kurvatur parabola

$$y = ax^2 + bx + c.$$

```
>maxima function f(x) &= a*x^2+b*x+c; $y=f(x)
```

```
          [0, 4.999958333473664e-5 r, 1.999933334222437e-4 r,
  4.499662510124569e-4 r, 7.998933390220841e-4 r,
  0.00124973960503371 7 r, 0.00179946006479581 r,
  0.00244899974672041 5 r, 0.00319829369738056 1 r,
  0.00404726698800572 7 r, 0.00499583472197417 9 r,
  0.00604390204330318 4 r, 0.00719136414613375 r, 0.00843810628521191 r,
  0.00978400378736277 2 r, 0.01122892206395776 r, 0.01277271662437307 r,
  0.01441523309043924 r, 0.01615630721187855 r, 0.01799576488272969 r,
  0.01993342215875837 r, 0.02196908527585173 r, 0.02410255066939448 r,
  0.02633360499462523 r, 0.02866202514797045 r, 0.03108757828935527 r,
  0.03361002186548678 r, 0.03622910363410947 r, 0.03894456168922911 r,
  0.04175612448730281 r, 0.04466351087439402 r, 0.04766643011428662 r,
  0.05076458191755917 r, 0.0539576564716131 r, 0.05724533447165381 r,
  0.06062728715262111 r, 0.06410317632206519 r, 0.06767265439396564 r,
  0.07133536442348987 r, 0.07509094014268702 r, 0.07893900599711501 r,
  0.08287917718339499 r, 0.08691105968769186 r, 0.09103425032511492 r,
  0.09524833678003664 r, 0.09955289764732322 r, 0.1039475024744748 r,
  0.1084317118046711 r, 0.113005077220716 r, 0.1176671413898787 r,
  0.1224174381096274 r, 0.1272554923542488 r, 0.1321808203223502 r,
  0.1371929294852391 r, 0.1422913186361759 r, 0.1474754779404944 r,
  0.152744888986584 r, 0.1580990248377314 r, 0.1635373500848132 r,
  0.1690593208998367 r, 0.1746643850903219 r, 0.1803519821545206 r,
  0.1861215433374662 r, 0.1919724916878484 r, 0.1979042421157076 r,
  0.2039162014509444 r, 0.2100077685026351 r, 0.216178334119151 r,
  0.2224272812490723 r, 0.2287539850028937 r, 0.2351578127155118 r,
  0.2416381240094921 r, 0.2481942708591053 r, 0.2548255976551299 r,
  0.2615314412704124 r, 0.2683111311261794 r, 0.2751639892590951 r,
  0.2820893303890569 r, 0.2890864619877229 r, 0.2961546843477643 r,
  0.3032932906528349 r, 0.3105015670482534 r, 0.3177787927123868 r,
  0.3251242399287333 r, 0.3325371741586922 r, 0.3400168541150183 r,
  0.3475625318359485 r, 0.3551734527599992 r, 0.3628488558014202 r,
  0.3705879734263036 r, 0.3783900317293359 r, 0.3862542505111889 r,
```

```
0.3941798433565377 r, 0.4021660177127022 r, 0.4102119749689023 r,
0.418316910536117 r, 0.4264800139275439 r, 0.4347004688396462 r,
0.4429774532337832 r, 0.451310139418413 r] =
                                2
[c, 2.7777500001498e-14 a r  + 1.66665833335744e-7 b r + c,
                               2
1.777706668053906e-12 a r  + 1.33330666692022e-6 b r + c,
                               2
2.024817758005038e-11 a r  + 4.499797504338432e-6 b r + c,
                               2
1.137595747549299e-10 a r  + 1.066581336583994e-5 b r + c,
                               2
4.339192840727639e-10 a r  + 2.083072932167196e-5 b r + c,
                              2
1.295533521972174e-9 a r  + 3.599352055540239e-5 b r + c,
                              2
3.266426827094104e-9 a r  + 5.71526624672386e-5 b r + c,
                              2
7.277118895509326e-9 a r  + 8.530603082730626e-5 b r + c,
                              2
1.475029730376073e-8 a r  + 1.214508019889565e-4 b r + c,
                              2
2.775001355397757e-8 a r  + 1.665833531718508e-4 b r + c,
                              2
4.915051879738995e-8 a r  + 2.216991628251896e-4 b r + c,
                             2
8.28246445511412e-8 a r  + 2.877927110806339e-4 b r + c,
                              2
1.33851622723744e-7 a r  + 3.658573803051457e-4 b r + c,
                              2
2.087442283111582e-7 a r  + 4.568853557635201e-4 b r + c,
                              2
3.156951172237287e-7 a r  + 5.618675264007778e-4 b r + c,
                             2
4.64842220857938e-7 a r  + 6.817933857540259e-4 b r + c,
                              2
6.685530482422835e-7 a r  + 8.176509330039827e-4 b r + c,
                              2
9.417277358666075e-7 a r  + 9.704265741758145e-4 b r + c,
                             2
1.30212067465563e-6 a r  + 0.001141105023499428 b r + c,
                              2
1.770680532972444e-6 a r  + 0.001330669204938795 b r + c,
                              2
2.371908484044149e-6 a r  + 0.001540100153900437 b r + c,
                              2
3.134234435790633e-6 a r  + 0.001770376919130678 b r + c,
                              2
4.090411050716832e-6 a r  + 0.002022476464811601 b r + c,
                              2
5.277925333300395e-6 a r  + 0.002297373572865413 b r + c,
                              2
6.739427552177103e-6 a r  + 0.002596040745477063 b r + c,
                              2
8.523177254399114e-6 a r  + 0.002919448107844891 b r + c,
                              2
```

```
1.068350611911921e-5 a r  + 0.003268563311168871 b r + c,
                        2
1.328129738824626e-5 a r  + 0.003644351435886262 b r + c,
                        2
1.638448160192355e-5 a r  + 0.004047774895164447 b r + c,
                        2
2.006854835710647e-5 a r  + 0.004479793338660443 b r + c,
                       2
2.44170737980647e-5 a r  + 0.0049413635565565 b r + c,
                        2
2.952226353832265e-5 a r  + 0.005433439383882244 b r + c,
                        2
3.548551070434468e-5 a r  + 0.005956971605131645 b r + c,
                        2
4.241796878224187e-5 a r  + 0.006512907859185624 b r + c,
                        2
5.044113893984222e-5 a r  + 0.007102192544548636 b r + c,
                        2
5.968747148772726e-5 a r  + 0.007725766724910044 b r + c,
                        2
7.030098113418114e-5 a r  + 0.00838456803503801 b r + c,
                        2
8.243787568058321e-5 a r  + 0.009079530587017326 b r + c,
                        2
9.626719779540763e-5 a r  + 0.009811584876838586 b r + c,
                       2
1.11971479496896e-4 a r  + 0.0105816576913495 b r + c,
                        2
1.297474089664522e-4 a r  + 0.01139067201557714 b r + c,
                        2
1.498065093069853e-4 a r  + 0.01223954694042984 b r + c,
                        2
1.723758288528179e-4 a r  + 0.01312919757078923 b r + c,
                        2
1.976986426302469e-4 a r  + 0.01406053493400045 b r + c,
                        2
2.260351645605837e-4 a r  + 0.01503446588876983 b r + c,
                        2
2.576632699903951e-4 a r  + 0.01605189303448024 b r + c,
                        2
2.928792281266932e-4 a r  + 0.01711371462093175 b r + c,
                        2
3.319984439480964e-4 a r  + 0.01822082445851714 b r + c,
                        2
3.753562091564763e-4 a r  + 0.01937411182884202 b r + c,
                        2
4.233084617271431e-4 a r  + 0.02057446139579705 b r + c,
                        2
4.762325536095718e-4 a r  + 0.02182275311709253 b r + c,
                       2
5.34528026124617e-4 a r  + 0.02311986215626333 b r + c,
                        2
5.986173925984417e-4 a r  + 0.02446665879515308 b r + c,
                        2
6.689469277678383e-4 a r  + 0.02586400834688696 b r + c,
                        2
```

```
7.459874634862211e-4 a r  + 0.02731277106934082 b r + c,
                       2
8.302351902545073e-4 a r  + 0.02881380207911666 b r + c,
                       2
9.222124640960191e-4 a r  + 0.03036795126603076 b r + c,
                       2
0.001022468618290102 a r  + 0.03197606320812652 b r + c,
                       2
0.001131580779474263 a r  + 0.0336389770872163 b r + c,
                       2
0.001250154687620788 a r  + 0.03535752660496472 b r + c,
                       2
0.001378825519389357 a r  + 0.03713253989951881 b r + c,
                       2
0.001518258714353595 a r  + 0.03896483946269502 b r + c,
                       2
0.001669150803595751 a r  + 0.0408552420577305 b r + c,
                       2
0.001832230240160423 a r  + 0.04280455863760801 b r + c,
                       2
0.002008258230854871 a r  + 0.04481359426396048 b r + c,
                       2
0.002198029568880921 a r  + 0.04688314802656623 b r + c,
                       2
0.002402373466780307 a r  + 0.04901401296344043 b r + c,
                       2
0.002622154389173151 a r  + 0.05120697598153157 b r + c,
                       2
0.002858272884767075 a r  + 0.05346281777803219 b r + c,
                       2
0.003111666417112067 a r  + 0.05578231276230905 b r + c,
                       2
0.003383310193575043 a r  + 0.05816622897846346 b r + c,
                       2
0.003674217992005929 a r  + 0.06061532802852698 b r + c,
                       2
0.003985442984566339 a r  + 0.0631303649963022 b r + c,
                       2
0.004318078558190487 a r  + 0.06571208837185505 b r + c,
                       2
0.004673259131147316 a r  + 0.06836123997666599 b r + c,
                       2
0.005052160965172387 a r  + 0.07107855488944881 b r + c,
                       2
0.005456002972637555 a r  + 0.07386476137264342 b r + c,
                       2
0.005886047518226416 a r  + 0.07672058079958999 b r + c,
                       2
0.006343601214583815 a r  + 0.07964672758239233 b r + c,
                       2
0.006830015711407966 a r  + 0.08264390910047736 b r + c,
                       2
0.007346688477454374 a r  + 0.0857128256298576 b r + c,
                       2
0.007895063574921807 a r  + 0.08885417027310427 b r + c,
                       2
```

```
0.008476632425691433 a r  + 0.09206862889003742 b r + c,
                         2
0.009092934568891969 a r  + 0.09535688002914089 b r + c,
                         2
0.009745558409264787 a r  + 0.0987195948597075 b r + c,
                        2
0.01043614195580549 a r  + 0.1021574371047232 b r + c,
                        2
0.01116637355015972 a r  + 0.1056710629744951 b r + c,
                        2
0.01193799258425414 a r  + 0.1092611211010309 b r + c,
                        2
0.01275279020664547 a r  + 0.1129282524731764 b r + c,
                        2
0.01361261001707348 a r  + 0.1166730903725168 b r + c,
                        2
0.01451934874870728 a r  + 0.1204962603100498 b r + c,
                        2
0.01547495693757671 a r  + 0.1243983799636342 b r + c,
                        2
0.01648143957868493 a r  + 0.1283800591162231 b r + c,
                        2
0.01754085676830185 a r  + 0.1324418995948859 b r + c,
                        2
0.01865532433194167 a r  + 0.1365844952106265 b r + c,
                        2
0.01982701443753252 a r  + 0.140808431699002 b r + c,
                        2
0.02105815619329058 a r  + 0.1451142866615502 b r + c,
                        2
0.02235103622981523 a r  + 0.1495026295080298 b r + c,
                        2
0.02370799926592746 a r  + 0.1539740213994798 b r + c]
```

```
>maxima function k(x) &= (diff(f(x),x,2))/(1+diff(f(x),x)^2)^(3/2); $'k(x)=k(x) // kelengk
```

```
        k([0, 1.66665833335744e-7 r, 1.33330666692022e-6 r,
4.499797504338432e-6 r, 1.066581336583994e-5 r,
2.083072932167196e-5 r, 3.599352055540239e-5 r,
5.71526624672386e-5 r, 8.530603082730626e-5 r,
1.214508019889565e-4 r, 1.665833531718508e-4 r,
2.216991628251896e-4 r, 2.877927110806339e-4 r,
3.658573803051457e-4 r, 4.5688535576352e-4 r, 5.618675264007778e-4 r,
6.817933857540259e-4 r, 8.176509330039827e-4 r,
9.704265741758145e-4 r, 0.001141105023499428 r,
0.001330669204938795 r, 0.001540100153900437 r,
0.001770376919130678 r, 0.002022476464811601 r,
0.002297373572865413 r, 0.002596040745477063 r,
0.002919448107844891 r, 0.003268563311168871 r,
0.003644351435886262 r, 0.004047774895164447 r,
0.004479793338660443 r, 0.0049413635565565 r, 0.005433439383882244 r,
0.005956971605131645 r, 0.006512907859185624 r,
0.007102192544548636 r, 0.007725766724910044 r,
```

0.00838456803503801 r, 0.009079530587017326 r,
0.009811584876838586 r, 0.0105816576913495 r, 0.01139067201557714 r,
0.01223954694042984 r, 0.01312919757078923 r, 0.01406053493400045 r,
0.01503446588876983 r, 0.01605189303448024 r, 0.01711371462093175 r,
0.01822082445851714 r, 0.01937411182884202 r, 0.02057446139579705 r,
0.02182275311709253 r, 0.02311986215626333 r, 0.02446665879515308 r,
0.02586400834688696 r, 0.02731277106934082 r, 0.02881380207911666 r,
0.03036795126603076 r, 0.03197606320812652 r, 0.0336389770872163 r,
0.03535752660496472 r, 0.03713253989951881 r, 0.03896483946269502 r,
0.0408552420577305 r, 0.04280455863760801 r, 0.04481359426396048 r,
0.04688314802656623 r, 0.04901401296344043 r, 0.05120697598153157 r,
0.05346281777803219 r, 0.05578231276230905 r, 0.05816622897846346 r,
0.06061532802852698 r, 0.0631303649963022 r, 0.06571208837185505 r,
0.06836123997666599 r, 0.07107855488944881 r, 0.07386476137264342 r,
0.07672058079958999 r, 0.07964672758239233 r, 0.08264390910047736 r,
0.0857128256298576 r, 0.08885417027310427 r, 0.09206862889003742 r,
0.09535688002914089 r, 0.0987195948597075 r, 0.1021574371047232 r,
0.1056710629744951 r, 0.1092611211010309 r, 0.1129282524731764 r,
0.1166730903725168 r, 0.1204962603100498 r, 0.1243983799636342 r,
0.1283800591162231 r, 0.1324418995948859 r, 0.1365844952106265 r,
0.140808431699002 r, 0.1451142866615502 r, 0.1495026295080298 r,
0.1539740213994798 r]) = k([0, 1.66665833335744e-7 r,
1.33330666692022e-6 r, 4.499797504338432e-6 r,
1.066581336583994e-5 r, 2.083072932167196e-5 r,
3.599352055540239e-5 r, 5.71526624672386e-5 r,
8.530603082730626e-5 r, 1.214508019889565e-4 r,
1.665833531718508e-4 r, 2.216991628251896e-4 r,
2.877927110806339e-4 r, 3.658573803051457e-4 r, 4.5688535576352e-4 r,
5.618675264007778e-4 r, 6.817933857540259e-4 r,
8.176509330039827e-4 r, 9.704265741758145e-4 r,
0.001141105023499428 r, 0.001330669204938795 r,
0.001540100153900437 r, 0.001770376919130678 r,
0.002022476464811601 r, 0.002297373572865413 r,
0.002596040745477063 r, 0.002919448107844891 r,
0.003268563311168871 r, 0.003644351435886262 r,
0.004047774895164447 r, 0.004479793338660443 r, 0.0049413635565565 r,
0.005433439383882244 r, 0.005956971605131645 r,
0.006512907859185624 r, 0.007102192544548636 r,
0.007725766724910044 r, 0.00838456803503801 r,
0.009079530587017326 r, 0.009811584876838586 r, 0.0105816576913495 r,
0.01139067201557714 r, 0.01223954694042984 r, 0.01312919757078923 r,
0.01406053493400045 r, 0.01503446588876983 r, 0.01605189303448024 r,
0.01711371462093175 r, 0.01822082445851714 r, 0.01937411182884202 r,
0.02057446139579705 r, 0.02182275311709253 r, 0.02311986215626333 r,
0.02446665879515308 r, 0.02586400834688696 r, 0.02731277106934082 r,
0.02881380207911666 r, 0.03036795126603076 r, 0.03197606320812652 r,
0.0336389770872163 r, 0.03535752660496472 r, 0.03713253989951881 r,
0.03896483946269502 r, 0.0408552420577305 r, 0.04280455863760801 r,
0.04481359426396048 r, 0.04688314802656623 r, 0.04901401296344043 r,
0.05120697598153157 r, 0.05346281777803219 r, 0.05578231276230905 r,
0.05816622897846346 r, 0.06061532802852698 r, 0.0631303649963022 r,
0.06571208837185505 r, 0.06836123997666599 r, 0.07107855488944881 r,
0.07386476137264342 r, 0.07672058079958999 r, 0.07964672758239233 r,
0.08264390910047736 r, 0.0857128256298576 r, 0.08885417027310427 r,
0.09206862889003742 r, 0.09535688002914089 r, 0.0987195948597075 r,
0.1021574371047232 r, 0.1056710629744951 r, 0.1092611211010309 r,

```
0.1129282524731764 r, 0.1166730903725168 r, 0.1204962603100498 r,
0.1243983799636342 r, 0.1283800591162231 r, 0.1324418995948859 r,
0.1365844952106265 r, 0.140808431699002 r, 0.1451142866615502 r,
0.1495026295080298 r, 0.1539740213994798 r])
```

```
>maxima function f(x) &= x^2+x+1; $y=f(x) // akan kita plot kelengkungan parabola untuk a=
```

```
        [0, 4.999958333473664e-5 r, 1.999933334222437e-4 r,
4.499662510124569e-4 r, 7.998933390220841e-4 r,
0.001249739605033717 r, 0.00179946006479581 r,
0.002448999746720415 r, 0.003198293697380561 r,
0.004047266988005727 r, 0.004995834721974179 r,
0.006043902043303184 r, 0.00719136414613375 r, 0.00843810628521191 r,
0.009784003787362772 r, 0.01122892206395776 r, 0.01277271662437307 r,
0.01441523309043924 r, 0.01615630721187855 r, 0.01799576488272969 r,
0.01993342215875837 r, 0.02196908527585173 r, 0.02410255066939448 r,
0.02633360499462523 r, 0.02866202514797045 r, 0.03108757828935527 r,
0.03361002186548678 r, 0.03622910363410947 r, 0.03894456168922911 r,
0.04175612448730281 r, 0.04466351087439402 r, 0.04766643011428662 r,
0.05076458191755917 r, 0.0539576564716131 r, 0.05724533447165381 r,
0.06062728715262111 r, 0.06410317632206519 r, 0.06767265439396564 r,
0.07133536442348987 r, 0.07509094014268702 r, 0.07893900599711501 r,
0.08287917718339499 r, 0.08691105968769186 r, 0.09103425032511492 r,
0.09524833678003664 r, 0.09955289764732322 r, 0.1039475024744748 r,
0.1084317118046711 r, 0.113005077220716 r, 0.1176671413898787 r,
0.1224174381096274 r, 0.1272554923542488 r, 0.1321808203223502 r,
0.1371929294852391 r, 0.1422913186361759 r, 0.1474754779404944 r,
0.152744888986584 r, 0.1580990248377314 r, 0.1635373500848132 r,
0.1690593208998367 r, 0.1746643850903219 r, 0.1803519821545206 r,
0.1861215433374662 r, 0.1919724916878484 r, 0.1979042421157076 r,
0.2039162014509444 r, 0.2100077685026351 r, 0.216178334119151 r,
0.2224272812490723 r, 0.2287539850028937 r, 0.2351578127155118 r,
0.2416381240094921 r, 0.2481942708591053 r, 0.2548255976551299 r,
0.2615314412704124 r, 0.2683111311261794 r, 0.2751639892590951 r,
0.2820893303890569 r, 0.2890864619877229 r, 0.2961546843477643 r,
0.3032932906528349 r, 0.3105015670482534 r, 0.3177787927123868 r,
0.3251242399287333 r, 0.3325371741586922 r, 0.3400168541150183 r,
0.3475625318359485 r, 0.3551734527599992 r, 0.3628488558014202 r,
0.3705879734263036 r, 0.3783900317293359 r, 0.3862542505111889 r,
0.3941798433565377 r, 0.4021660177127022 r, 0.4102119749689023 r,
0.418316910536117 r, 0.4264800139275439 r, 0.4347004688396462 r,
0.4429774532337832 r, 0.451310139418413 r] =
                              2
[c, 2.7777500001498e-14 a r  + 1.66665833335744e-7 b r + c,
                             2
1.777706668053906e-12 a r  + 1.33330666692022e-6 b r + c,
                             2
2.024817758005038e-11 a r  + 4.499797504338432e-6 b r + c,
                             2
1.137595747549299e-10 a r  + 1.066581336583994e-5 b r + c,
                             2
4.339192840727639e-10 a r  + 2.083072932167196e-5 b r + c,
                            2
```

```
1.295533521972174e-9 a r  + 3.599352055540239e-5 b r + c,
                        2
3.266426827094104e-9 a r  + 5.71526624672386e-5 b r + c,
                        2
7.277118895509326e-9 a r  + 8.530603082730626e-5 b r + c,
                        2
1.475029730376073e-8 a r  + 1.214508019889565e-4 b r + c,
                        2
2.775001355397757e-8 a r  + 1.665833531718508e-4 b r + c,
                        2
4.915051879738995e-8 a r  + 2.216991628251896e-4 b r + c,
                        2
8.28246445511412e-8 a r  + 2.877927110806339e-4 b r + c,
                       2
1.33851622723744e-7 a r  + 3.658573803051457e-4 b r + c,
                        2
2.087442283111582e-7 a r  + 4.568853557635201e-4 b r + c,
                        2
3.156951172237287e-7 a r  + 5.618675264007778e-4 b r + c,
                       2
4.64842220857938e-7 a r  + 6.817933857540259e-4 b r + c,
                        2
6.685530482422835e-7 a r  + 8.176509330039827e-4 b r + c,
                        2
9.417277358666075e-7 a r  + 9.704265741758145e-4 b r + c,
                       2
1.30212067465563e-6 a r  + 0.001141105023499428 b r + c,
                        2
1.770680532972444e-6 a r  + 0.001330669204938795 b r + c,
                        2
2.371908484044149e-6 a r  + 0.001540100153900437 b r + c,
                        2
3.134234435790633e-6 a r  + 0.001770376919130678 b r + c,
                        2
4.090411050716832e-6 a r  + 0.002022476464811601 b r + c,
                        2
5.277925333300395e-6 a r  + 0.002297373572865413 b r + c,
                        2
6.739427552177103e-6 a r  + 0.002596040745477063 b r + c,
                        2
8.523177254399114e-6 a r  + 0.002919448107844891 b r + c,
                        2
1.068350611911921e-5 a r  + 0.003268563311168871 b r + c,
                        2
1.328129738824626e-5 a r  + 0.003644351435886262 b r + c,
                        2
1.638448160192355e-5 a r  + 0.004047774895164447 b r + c,
                        2
2.006854835710647e-5 a r  + 0.004479793338660443 b r + c,
                       2
2.44170737980647e-5 a r  + 0.0049413635565565 b r + c,
                        2
2.952226353832265e-5 a r  + 0.005433439383882244 b r + c,
                        2
3.548551070434468e-5 a r  + 0.005956971605131645 b r + c,
                        2
```

```
4.241796878224187e-5 a r  + 0.006512907859185624 b r + c,
                       2
5.044113893984222e-5 a r  + 0.007102192544548636 b r + c,
                       2
5.968747148772726e-5 a r  + 0.007725766724910044 b r + c,
                       2
7.030098113418114e-5 a r  + 0.00838456803503801 b r + c,
                       2
8.243787568058321e-5 a r  + 0.009079530587017326 b r + c,
                       2
9.626719779540763e-5 a r  + 0.009811584876838586 b r + c,
                      2
1.11971479496896e-4 a r  + 0.0105816576913495 b r + c,
                       2
1.297474089664522e-4 a r  + 0.01139067201557714 b r + c,
                       2
1.498065093069853e-4 a r  + 0.01223954694042984 b r + c,
                       2
1.723758288528179e-4 a r  + 0.01312919757078923 b r + c,
                       2
1.976986426302469e-4 a r  + 0.01406053493400045 b r + c,
                       2
2.260351645605837e-4 a r  + 0.01503446588876983 b r + c,
                       2
2.576632699903951e-4 a r  + 0.01605189303448024 b r + c,
                       2
2.928792281266932e-4 a r  + 0.01711371462093175 b r + c,
                       2
3.319984439480964e-4 a r  + 0.01822082445851714 b r + c,
                       2
3.753562091564763e-4 a r  + 0.01937411182884202 b r + c,
                       2
4.233084617271431e-4 a r  + 0.02057446139579705 b r + c,
                       2
4.762325536095718e-4 a r  + 0.02182275311709253 b r + c,
                      2
5.34528026124617e-4 a r  + 0.02311986215626333 b r + c,
                       2
5.986173925984417e-4 a r  + 0.02446665879515308 b r + c,
                       2
6.689469277678383e-4 a r  + 0.02586400834688696 b r + c,
                       2
7.459874634862211e-4 a r  + 0.02731277106934082 b r + c,
                       2
8.302351902545073e-4 a r  + 0.02881380207911666 b r + c,
                       2
9.222124640960191e-4 a r  + 0.03036795126603076 b r + c,
                       2
0.001022468618290102 a r  + 0.03197606320812652 b r + c,
                       2
0.001131580779474263 a r  + 0.0336389770872163 b r + c,
                       2
0.001250154687620788 a r  + 0.03535752660496472 b r + c,
                       2
0.001378825519389357 a r  + 0.03713253989951881 b r + c,
                       2
```

```
0.001518258714353595 a r  + 0.03896483946269502 b r + c,
                        2
0.001669150803595751 a r  + 0.0408552420577305 b r + c,
                        2
0.001832230240160423 a r  + 0.04280455863760801 b r + c,
                        2
0.002008258230854871 a r  + 0.04481359426396048 b r + c,
                        2
0.002198029568880921 a r  + 0.04688314802656623 b r + c,
                        2
0.002402373466780307 a r  + 0.04901401296344043 b r + c,
                        2
0.002622154389173151 a r  + 0.05120697598153157 b r + c,
                        2
0.002858272884767075 a r  + 0.05346281777803219 b r + c,
                        2
0.003111666417112067 a r  + 0.05578231276230905 b r + c,
                        2
0.003383310193575043 a r  + 0.05816622897846346 b r + c,
                        2
0.003674217992005929 a r  + 0.06061532802852698 b r + c,
                        2
0.003985442984566339 a r  + 0.0631303649963022 b r + c,
                        2
0.004318078558190487 a r  + 0.06571208837185505 b r + c,
                        2
0.004673259131147316 a r  + 0.06836123997666599 b r + c,
                        2
0.005052160965172387 a r  + 0.07107855488944881 b r + c,
                        2
0.005456002972637555 a r  + 0.07386476137264342 b r + c,
                        2
0.005886047518226416 a r  + 0.07672058079958999 b r + c,
                        2
0.006343601214583815 a r  + 0.07964672758239233 b r + c,
                        2
0.006830015711407966 a r  + 0.08264390910047736 b r + c,
                        2
0.007346688477454374 a r  + 0.0857128256298576 b r + c,
                        2
0.007895063574921807 a r  + 0.08885417027310427 b r + c,
                        2
0.008476632425691433 a r  + 0.09206862889003742 b r + c,
                        2
0.009092934568891969 a r  + 0.09535688002914089 b r + c,
                        2
0.009745558409264787 a r  + 0.0987195948597075 b r + c,
                        2
0.01043614195580549 a r  + 0.1021574371047232 b r + c,
                        2
0.01116637355015972 a r  + 0.1056710629744951 b r + c,
                        2
0.01193799258425414 a r  + 0.1092611211010309 b r + c,
                        2
0.01275279020664547 a r  + 0.1129282524731764 b r + c,
                        2
```

```
0.01361261001707348 a r  + 0.1166730903725168 b r + c,
                        2
0.01451934874870728 a r  + 0.1204962603100498 b r + c,
                        2
0.01547495693757671 a r  + 0.1243983799636342 b r + c,
                        2
0.01648143957868493 a r  + 0.1283800591162231 b r + c,
                        2
0.01754085676830185 a r  + 0.1324418995948859 b r + c,
                        2
0.01865532433194167 a r  + 0.1365844952106265 b r + c,
                        2
0.01982701443753252 a r  + 0.140808431699002 b r + c,
                        2
0.02105815619329058 a r  + 0.1451142866615502 b r + c,
                        2
0.02235103622981523 a r  + 0.1495026295080298 b r + c,
                        2
0.02370799926592746 a r  + 0.1539740213994798 b r + c]
```

```
>maxima function k(x) &= (diff(f(x),x,2))/(1+diff(f(x),x)^2)^(3/2); $'k(x)=k(x) // kelengk
```

```
        k([0, 1.66665833335744e-7 r, 1.33330666692022e-6 r,
4.499797504338432e-6 r, 1.066581336583994e-5 r,
2.083072932167196e-5 r, 3.599352055540239e-5 r,
5.71526624672386e-5 r, 8.530603082730626e-5 r,
1.214508019889565e-4 r, 1.665833531718508e-4 r,
2.216991628251896e-4 r, 2.877927110806339e-4 r,
3.658573803051457e-4 r, 4.5688535576352e-4 r, 5.618675264007778e-4 r,
6.817933857540259e-4 r, 8.176509330039827e-4 r,
9.704265741758145e-4 r, 0.001141105023499428 r,
0.001330669204938795 r, 0.001540100153900437 r,
0.001770376919130678 r, 0.002022476464811601 r,
0.002297373572865413 r, 0.002596040745477063 r,
0.002919448107844891 r, 0.003268563311168871 r,
0.003644351435886262 r, 0.004047774895164447 r,
0.004479793338660443 r, 0.0049413635565565 r, 0.005433439383882244 r,
0.005956971605131645 r, 0.006512907859185624 r,
0.007102192544548636 r, 0.007725766724910044 r,
0.00838456803503801 r, 0.009079530587017326 r,
0.009811584876838586 r, 0.0105816576913495 r, 0.01139067201557714 r,
0.01223954694042984 r, 0.01312919757078923 r, 0.01406053493400045 r,
0.01503446588876983 r, 0.01605189303448024 r, 0.01711371462093175 r,
0.01822082445851714 r, 0.01937411182884202 r, 0.02057446139579705 r,
0.02182275311709253 r, 0.02311986215626333 r, 0.02446665879515308 r,
0.02586400834688696 r, 0.02731277106934082 r, 0.02881380207911666 r,
0.03036795126603076 r, 0.03197606320812652 r, 0.0336389770872163 r,
0.03535752660496472 r, 0.03713253989951881 r, 0.03896483946269502 r,
0.0408552420577305 r, 0.04280455863760801 r, 0.04481359426396048 r,
0.04688314802656623 r, 0.04901401296344043 r, 0.05120697598153157 r,
0.05346281777803219 r, 0.05578231276230905 r, 0.05816622897846346 r,
0.06061532802852698 r, 0.0631303649963022 r, 0.06571208837185505 r,
0.06836123997666599 r, 0.07107855488944881 r, 0.07386476137264342 r,
```

```
0.07672058079958999 r, 0.07964672758239233 r, 0.08264390910047736 r,
0.0857128256298576 r, 0.08885417027310427 r, 0.09206862889003742 r,
0.09535688002914089 r, 0.0987195948597075 r, 0.1021574371047232 r,
0.1056710629744951 r, 0.1092611211010309 r, 0.1129282524731764 r,
0.1166730903725168 r, 0.1204962603100498 r, 0.1243983799636342 r,
0.1283800591162231 r, 0.1324418995948859 r, 0.1365844952106265 r,
0.140808431699002 r, 0.1451142866615502 r, 0.1495026295080298 r,
0.1539740213994798 r]) = k([0, 1.66665833335744e-7 r,
1.33330666692022e-6 r, 4.499797504338432e-6 r,
1.066581336583994e-5 r, 2.083072932167196e-5 r,
3.599352055540239e-5 r, 5.71526624672386e-5 r,
8.530603082730626e-5 r, 1.214508019889565e-4 r,
1.665833531718508e-4 r, 2.216991628251896e-4 r,
2.877927110806339e-4 r, 3.658573803051457e-4 r, 4.5688535576352e-4 r,
5.618675264007778e-4 r, 6.817933857540259e-4 r,
8.176509330039827e-4 r, 9.704265741758145e-4 r,
0.001141105023499428 r, 0.001330669204938795 r,
0.001540100153900437 r, 0.001770376919130678 r,
0.002022476464811601 r, 0.002297373572865413 r,
0.002596040745477063 r, 0.002919448107844891 r,
0.003268563311168871 r, 0.003644351435886262 r,
0.004047774895164447 r, 0.004479793338660443 r, 0.0049413635565565 r,
0.005433439383882244 r, 0.005956971605131645 r,
0.006512907859185624 r, 0.007102192544548636 r,
0.007725766724910044 r, 0.00838456803503801 r,
0.009079530587017326 r, 0.009811584876838586 r, 0.0105816576913495 r,
0.01139067201557714 r, 0.01223954694042984 r, 0.01312919757078923 r,
0.01406053493400045 r, 0.01503446588876983 r, 0.01605189303448024 r,
0.01711371462093175 r, 0.01822082445851714 r, 0.01937411182884202 r,
0.02057446139579705 r, 0.02182275311709253 r, 0.02311986215626333 r,
0.02446665879515308 r, 0.02586400834688696 r, 0.02731277106934082 r,
0.02881380207911666 r, 0.03036795126603076 r, 0.03197606320812652 r,
0.0336389770872163 r, 0.03535752660496472 r, 0.03713253989951881 r,
0.03896483946269502 r, 0.0408552420577305 r, 0.04280455863760801 r,
0.04481359426396048 r, 0.04688314802656623 r, 0.04901401296344043 r,
0.05120697598153157 r, 0.05346281777803219 r, 0.05578231276230905 r,
0.05816622897846346 r, 0.06061532802852698 r, 0.0631303649963022 r,
0.06571208837185505 r, 0.06836123997666599 r, 0.07107855488944881 r,
0.07386476137264342 r, 0.07672058079958999 r, 0.07964672758239233 r,
0.08264390910047736 r, 0.0857128256298576 r, 0.08885417027310427 r,
0.09206862889003742 r, 0.09535688002914089 r, 0.0987195948597075 r,
0.1021574371047232 r, 0.1056710629744951 r, 0.1092611211010309 r,
0.1129282524731764 r, 0.1166730903725168 r, 0.1204962603100498 r,
0.1243983799636342 r, 0.1283800591162231 r, 0.1324418995948859 r,
0.1365844952106265 r, 0.140808431699002 r, 0.1451142866615502 r,
0.1495026295080298 r, 0.1539740213994798 r])
```

Berikut kita gambar parabola tersebut beserta kurva kelengkungan, kurva jari-jari kelengkungan dan salah satu lingkaran oskulasi di titik puncak parabola. Perhatikan, puncak parabola dan jari-jari lingkaran oskulasi di puncak parabola adalah

$$(-1/2, 3/4),\ 1/k(2) = 1/2,$$

sehingga pusat lingkaran oskulasi adalah (-1/2, 5/4).

```
>plot2d(["f(x)", "k(x)"],-2,1, color=[blue,red]): plot2d("1/k(x)",-1.5,1,color=green,>add)
>plot2d("-1/2+1/k(-1/2)*cos(x)","5/4+1/k(-1/2)*sin(x)",xmin=0,xmax=2pi,>add,color=blue):
```

```
Variable c not found!
Use global or local variables defined in function f.
f:
    useglobal; return [c,2.77775e-14*a*r^2+1.66666e-07*b*r+c,1.77771e-12 ...
Error in expression: f(x)
%ploteval:
    y0=f$(x[1],args());
adaptiveevalone:
    s=%ploteval(g$,t;args());
Try "trace errors" to inspect local variables after errors.
plot2d:
    dw/n,dw/n^2,dw/n,auto;args());
```

Untuk kurva yang dinyatakan dengan fungsi implisit

$$F(x,y)=0$$

dengan turunan-turunan parsial

$$F_x=\frac{\partial F}{\partial x},\ F_y=\frac{\partial F}{\partial y},\ F_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right),\ F_{xx}=\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right),\ F_{yy}=\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right),$$

berlaku

$$F_x dx+F_y dy=0 \text{ atau } \frac{dy}{dx}=-\frac{F_x}{F_y},$$

sehingga kurvaturnya adalah

$$\kappa=\frac{F_y^2F_{xx}-2F_xF_yF_{xy}+F_x^2F_{yy}}{\left(F_x^2+F_y^2\right)^{3/2}}.$$

(Silakan Anda turunkan sendiri!)

Contoh 1:
Parabola

$$y=ax^2+bx+c$$

dapat dinyatakan ke dalam persamaan implisit

$$ax^2+bx+c-y=0.$$

```
>maxima function F(x,y) &=a*x^2+b*x+c-y; $F(x,y)
```

```
                                    2
        [c, 2.7777500001498e-14 a r  + 1.66665833335744e-7 b r
                                                          2
  - 4.999958333473664e-5 r + c, 1.777706668053906e-12 a r
```

```
 + 1.33330666692022e-6 b r - 1.999933334222437e-4 r + c,
                          2
2.024817758005038e-11 a r  + 4.499797504338432e-6 b r
                                                        2
 - 4.499662510124569e-4 r + c, 1.137595747549299e-10 a r
 + 1.066581336583994e-5 b r - 7.998933390220841e-4 r + c,
                          2
4.339192840727639e-10 a r  + 2.083072932167196e-5 b r
                                                      2
 - 0.001249739605033717 r + c, 1.295533521972174e-9 a r
 + 3.599352055540239e-5 b r - 0.00179946006479581 r + c,
                         2
3.266426827094104e-9 a r  + 5.71526624672386e-5 b r
                                                       2
 - 0.002448999746720415 r + c, 7.277118895509326e-9 a r
 + 8.530603082730626e-5 b r - 0.003198293697380561 r + c,
                         2
1.475029730376073e-8 a r  + 1.214508019889565e-4 b r
                                                       2
 - 0.004047266988005727 r + c, 2.775001355397757e-8 a r
 + 1.665833531718508e-4 b r - 0.004995834721974179 r + c,
                         2
4.915051879738995e-8 a r  + 2.216991628251896e-4 b r
                                                      2
 - 0.006043902043303184 r + c, 8.28246445511412e-8 a r
 + 2.877927110806339e-4 b r - 0.00719136414613375 r + c,
                        2
1.33851622723744e-7 a r  + 3.658573803051457e-4 b r
                                                      2
 - 0.00843810628521191 r + c, 2.087442283111582e-7 a r
 + 4.568853557635201e-4 b r - 0.009784003787362772 r + c,
                         2
3.156951172237287e-7 a r  + 5.618675264007778e-4 b r
                                                       2
 - 0.01122892206395776 r + c, 4.64842220857938e-7 a r
 + 6.817933857540259e-4 b r - 0.01277271662437307 r + c,
                         2
6.685530482422835e-7 a r  + 8.176509330039827e-4 b r
                                                      2
 - 0.01441523309043924 r + c, 9.417277358666075e-7 a r
 + 9.704265741758145e-4 b r - 0.01615630721187855 r + c,
                        2
1.30212067465563e-6 a r  + 0.001141105023499428 b r
                                                      2
 - 0.01799576488272969 r + c, 1.770680532972444e-6 a r
 + 0.001330669204938795 b r - 0.01993342215875837 r + c,
                         2
2.371908484044149e-6 a r  + 0.001540100153900437 b r
                                                      2
 - 0.02196908527585173 r + c, 3.134234435790633e-6 a r
 + 0.001770376919130678 b r - 0.02410255066939448 r + c,
                         2
4.090411050716832e-6 a r  + 0.002022476464811601 b r
                                                      2
 - 0.02633360499462523 r + c, 5.277925333300395e-6 a r
 + 0.002297373572865413 b r - 0.02866202514797045 r + c,
```

```
                          2
6.739427552177103e-6 a r  + 0.002596040745477063 b r
                                                      2
 - 0.03108757828935527 r + c, 8.523177254399114e-6 a r
 + 0.002919448107844891 b r - 0.03361002186548678 r + c,
                          2
1.068350611911921e-5 a r  + 0.003268563311168871 b r
                                                      2
 - 0.03622910363410947 r + c, 1.328129738824626e-5 a r
 + 0.003644351435886262 b r - 0.03894456168922911 r + c,
                          2
1.638448160192355e-5 a r  + 0.004047774895164447 b r
                                                      2
 - 0.04175612448730281 r + c, 2.006854835710647e-5 a r
 + 0.004479793338660443 b r - 0.04466351087439402 r + c,
                         2
2.44170737980647e-5 a r  + 0.0049413635565565 b r
                                                      2
 - 0.04766643011428662 r + c, 2.952226353832265e-5 a r
 + 0.005433439383882244 b r - 0.05076458191755917 r + c,
                          2
3.548551070434468e-5 a r  + 0.005956971605131645 b r
                                                     2
 - 0.0539576564716131 r + c, 4.241796878224187e-5 a r
 + 0.006512907859185624 b r - 0.05724533447165381 r + c,
                          2
5.044113893984222e-5 a r  + 0.007102192544548636 b r
                                                      2
 - 0.06062728715262111 r + c, 5.968747148772726e-5 a r
 + 0.007725766724910044 b r - 0.06410317632206519 r + c,
                          2
7.030098113418114e-5 a r  + 0.00838456803503801 b r
                                                      2
 - 0.06767265439396564 r + c, 8.243787568058321e-5 a r
 + 0.009079530587017326 b r - 0.07133536442348987 r + c,
                          2
9.626719779540763e-5 a r  + 0.009811584876838586 b r
                                                     2
 - 0.07509094014268702 r + c, 1.11971479496896e-4 a r
 + 0.0105816576913495 b r - 0.07893900599711501 r + c,
                          2
1.297474089664522e-4 a r  + 0.01139067201557714 b r
                                                      2
 - 0.08287917718339499 r + c, 1.498065093069853e-4 a r
 + 0.01223954694042984 b r - 0.08691105968769186 r + c,
                          2
1.723758288528179e-4 a r  + 0.01312919757078923 b r
                                                      2
 - 0.09103425032511492 r + c, 1.976986426302469e-4 a r
 + 0.01406053493400045 b r - 0.09524833678003664 r + c,
                          2
2.260351645605837e-4 a r  + 0.01503446588876983 b r
                                                      2
 - 0.09955289764732322 r + c, 2.576632699903951e-4 a r
 + 0.01605189303448024 b r - 0.1039475024744748 r + c,
                          2
```

```
2.928792281266932e-4 a r  + 0.01711371462093175 b r
                                                     2
 - 0.1084317118046711 r + c, 3.319984439480964e-4 a r
 + 0.01822082445851714 b r - 0.113005077220716 r + c,
                          2
3.753562091564763e-4 a r  + 0.01937411182884202 b r
                                                     2
 - 0.1176671413898787 r + c, 4.233084617271431e-4 a r
 + 0.02057446139579705 b r - 0.1224174381096274 r + c,
                          2
4.762325536095718e-4 a r  + 0.02182275311709253 b r
                                                     2
 - 0.1272554923542488 r + c, 5.34528026124617e-4 a r
 + 0.02311986215626333 b r - 0.1321808203223502 r + c,
                          2
5.986173925984417e-4 a r  + 0.02446665879515308 b r
                                                     2
 - 0.1371929294852391 r + c, 6.689469277678383e-4 a r
 + 0.02586400834688696 b r - 0.1422913186361759 r + c,
                          2
7.459874634862211e-4 a r  + 0.02731277106934082 b r
                                                     2
 - 0.1474754779404944 r + c, 8.302351902545073e-4 a r
 + 0.02881380207911666 b r - 0.152744888986584 r + c,
                          2
9.222124640960191e-4 a r  + 0.03036795126603076 b r
                                                     2
 - 0.1580990248377314 r + c, 0.001022468618290102 a r
 + 0.03197606320812652 b r - 0.1635373500848132 r + c,
                          2
0.001131580779474263 a r  + 0.0336389770872163 b r
                                                     2
 - 0.1690593208998367 r + c, 0.001250154687620788 a r
 + 0.03535752660496472 b r - 0.1746643850903219 r + c,
                          2
0.001378825519389357 a r  + 0.03713253989951881 b r
                                                     2
 - 0.1803519821545206 r + c, 0.001518258714353595 a r
 + 0.03896483946269502 b r - 0.1861215433374662 r + c,
                          2
0.001669150803595751 a r  + 0.0408552420577305 b r
                                                     2
 - 0.1919724916878484 r + c, 0.001832230240160423 a r
 + 0.04280455863760801 b r - 0.1979042421157076 r + c,
                          2
0.002008258230854871 a r  + 0.04481359426396048 b r
                                                     2
 - 0.2039162014509444 r + c, 0.002198029568880921 a r
 + 0.04688314802656623 b r - 0.2100077685026351 r + c,
                          2
0.002402373466780307 a r  + 0.04901401296344043 b r
                                                     2
 - 0.216178334119151 r + c, 0.002622154389173151 a r
 + 0.05120697598153157 b r - 0.2224272812490723 r + c,
                          2
0.002858272884767075 a r  + 0.05346281777803219 b r
```

```
                                                                    2
 - 0.2287539850028937 r + c, 0.003111666417112067 a r
 + 0.05578231276230905 b r - 0.2351578127155118 r + c,
                          2
0.003383310193575043 a r  + 0.05816622897846346 b r
                                                                    2
 - 0.2416381240094921 r + c, 0.003674217992005929 a r
 + 0.06061532802852698 b r - 0.2481942708591053 r + c,
                          2
0.003985442984566339 a r  + 0.0631303649963022 b r
                                                                    2
 - 0.2548255976551299 r + c, 0.004318078558190487 a r
 + 0.06571208837185505 b r - 0.2615314412704124 r + c,
                          2
0.004673259131147316 a r  + 0.06836123997666599 b r
                                                                    2
 - 0.2683111311261794 r + c, 0.005052160965172387 a r
 + 0.07107855488944881 b r - 0.2751639892590951 r + c,
                          2
0.005456002972637555 a r  + 0.07386476137264342 b r
                                                                    2
 - 0.2820893303890569 r + c, 0.005886047518226416 a r
 + 0.07672058079958999 b r - 0.2890864619877229 r + c,
                          2
0.006343601214583815 a r  + 0.07964672758239233 b r
                                                                    2
 - 0.2961546843477643 r + c, 0.006830015711407966 a r
 + 0.08264390910047736 b r - 0.3032932906528349 r + c,
                          2
0.007346688477454374 a r  + 0.0857128256298576 b r
                                                                    2
 - 0.3105015670482534 r + c, 0.007895063574921807 a r
 + 0.08885417027310427 b r - 0.3177787927123868 r + c,
                          2
0.008476632425691433 a r  + 0.09206862889003742 b r
                                                                    2
 - 0.3251242399287333 r + c, 0.009092934568891969 a r
 + 0.09535688002914089 b r - 0.3325371741586922 r + c,
                          2
0.009745558409264787 a r  + 0.0987195948597075 b r
                                                                   2
 - 0.3400168541150183 r + c, 0.01043614195580549 a r
 + 0.1021574371047232 b r - 0.3475625318359485 r + c,
                         2
0.01116637355015972 a r  + 0.1056710629744951 b r
                                                                   2
 - 0.3551734527599992 r + c, 0.01193799258425414 a r
 + 0.1092611211010309 b r - 0.3628488558014202 r + c,
                         2
0.01275279020664547 a r  + 0.1129282524731764 b r
                                                                   2
 - 0.3705879734263036 r + c, 0.01361261001707348 a r
 + 0.1166730903725168 b r - 0.3783900317293359 r + c,
                         2
0.01451934874870728 a r  + 0.1204962603100498 b r
                                                                   2
```

```
 - 0.3862542505111889 r + c, 0.01547495693757671 a r
 + 0.1243983799636342 b r - 0.3941798433565377 r + c,
                       2
0.01648143957868493 a r  + 0.1283800591162231 b r
                                                    2
 - 0.4021660177127022 r + c, 0.01754085676830185 a r
 + 0.1324418995948859 b r - 0.4102119749689023 r + c,
                       2
0.01865532433194167 a r  + 0.1365844952106265 b r
                                                   2
 - 0.418316910536117 r + c, 0.01982701443753252 a r
 + 0.140808431699002 b r - 0.4264800139275439 r + c,
                       2
0.02105815619329058 a r  + 0.1451142866615502 b r
                                                    2
 - 0.4347004688396462 r + c, 0.02235103622981523 a r
 + 0.1495026295080298 b r - 0.4429774532337832 r + c,
                       2
0.02370799926592746 a r  + 0.1539740213994798 b r
 - 0.451310139418413 r + c]
```

```
>maxima Fx &= diff(F(x,y),x), Fxx &=diff(F(x,y),x,2), Fy &=diff(F(x,y),y), Fxy &=diff(diff
```

```
incorrect syntax: & is not an infix operator
Fx &=
   ^


incorrect syntax: & is not an infix operator
 Fxx &=
     ^


incorrect syntax: & is not an infix operator
 Fy &=
    ^


incorrect syntax: & is not an infix operator
 Fxy &=
     ^


incorrect syntax: & is not an infix operator
 Fyy &=
     ^
```

```
>maxima function k(x) &= (Fy^2*Fxx-2*Fx*Fy*Fxy+Fx^2*Fyy)/(Fx^2+Fy^2)^(3/2); $'k(x)=k(x) //
```

```
        k([0, 1.66665833335744e-7 r, 1.33330666692022e-6 r,
4.499797504338432e-6 r, 1.066581336583994e-5 r,
2.083072932167196e-5 r, 3.599352055540239e-5 r,
5.71526624672386e-5 r, 8.530603082730626e-5 r,
1.214508019889565e-4 r, 1.665833531718508e-4 r,
2.216991628251896e-4 r, 2.877927110806339e-4 r,
3.658573803051457e-4 r, 4.5688535576352e-4 r, 5.618675264007778e-4 r,
6.817933857540259e-4 r, 8.176509330039827e-4 r,
9.704265741758145e-4 r, 0.001141105023499428 r,
0.001330669204938795 r, 0.001540100153900437 r,
0.001770376919130678 r, 0.002022476464811601 r,
0.002297373572865413 r, 0.002596040745477063 r,
0.002919448107844891 r, 0.003268563311168871 r,
0.003644351435886262 r, 0.004047774895164447 r,
0.004479793338660443 r, 0.0049413635565565 r, 0.005433439383882244 r,
0.005956971605131645 r, 0.006512907859185624 r,
0.007102192544548636 r, 0.007725766724910044 r,
0.00838456803503801 r, 0.009079530587017326 r,
0.009811584876838586 r, 0.0105816576913495 r, 0.01139067201557714 r,
0.01223954694042984 r, 0.01312919757078923 r, 0.01406053493400045 r,
0.01503446588876983 r, 0.01605189303448024 r, 0.01711371462093175 r,
0.01822082445851714 r, 0.01937411182884202 r, 0.02057446139579705 r,
0.02182275311709253 r, 0.02311986215626333 r, 0.02446665879515308 r,
0.02586400834688696 r, 0.02731277106934082 r, 0.02881380207911666 r,
0.03036795126603076 r, 0.03197606320812652 r, 0.0336389770872163 r,
0.03535752660496472 r, 0.03713253989951881 r, 0.03896483946269502 r,
0.0408552420577305 r, 0.04280455863760801 r, 0.04481359426396048 r,
0.04688314802656623 r, 0.04901401296344043 r, 0.05120697598153157 r,
0.05346281777803219 r, 0.05578231276230905 r, 0.05816622897846346 r,
0.06061532802852698 r, 0.0631303649963022 r, 0.06571208837185505 r,
0.06836123997666599 r, 0.07107855488944881 r, 0.07386476137264342 r,
0.07672058079958999 r, 0.07964672758239233 r, 0.08264390910047736 r,
0.0857128256298576 r, 0.08885417027310427 r, 0.09206862889003742 r,
0.09535688002914089 r, 0.0987195948597075 r, 0.1021574371047232 r,
0.1056710629744951 r, 0.1092611211010309 r, 0.1129282524731764 r,
0.1166730903725168 r, 0.1204962603100498 r, 0.1243983799636342 r,
0.1283800591162231 r, 0.1324418995948859 r, 0.1365844952106265 r,
0.140808431699002 r, 0.1451142866615502 r, 0.1495026295080298 r,
0.1539740213994798 r]) = k([0, 1.66665833335744e-7 r,
1.33330666692022e-6 r, 4.499797504338432e-6 r,
1.066581336583994e-5 r, 2.083072932167196e-5 r,
3.599352055540239e-5 r, 5.71526624672386e-5 r,
8.530603082730626e-5 r, 1.214508019889565e-4 r,
1.665833531718508e-4 r, 2.216991628251896e-4 r,
2.877927110806339e-4 r, 3.658573803051457e-4 r, 4.5688535576352e-4 r,
5.618675264007778e-4 r, 6.817933857540259e-4 r,
8.176509330039827e-4 r, 9.704265741758145e-4 r,
0.001141105023499428 r, 0.001330669204938795 r,
0.001540100153900437 r, 0.001770376919130678 r,
0.002022476464811601 r, 0.002297373572865413 r,
0.002596040745477063 r, 0.002919448107844891 r,
0.003268563311168871 r, 0.003644351435886262 r,
0.004047774895164447 r, 0.004479793338660443 r, 0.0049413635565565 r,
0.005433439383882244 r, 0.005956971605131645 r,
0.006512907859185624 r, 0.007102192544548636 r,
0.007725766724910044 r, 0.00838456803503801 r,
```

```
0.009079530587017326 r, 0.009811584876838586 r, 0.0105816576913495 r,
0.01139067201557714 r, 0.01223954694042984 r, 0.01312919757078923 r,
0.01406053493400045 r, 0.01503446588876983 r, 0.01605189303448024 r,
0.01711371462093175 r, 0.01822082445851714 r, 0.01937411182884202 r,
0.02057446139579705 r, 0.02182275311709253 r, 0.02311986215626333 r,
0.02446665879515308 r, 0.02586400834688696 r, 0.02731277106934082 r,
0.02881380207911666 r, 0.03036795126603076 r, 0.03197606320812652 r,
0.0336389770872163 r, 0.03535752660496472 r, 0.03713253989951881 r,
0.03896483946269502 r, 0.0408552420577305 r, 0.04280455863760801 r,
0.04481359426396048 r, 0.04688314802656623 r, 0.04901401296344043 r,
0.05120697598153157 r, 0.05346281777803219 r, 0.05578231276230905 r,
0.05816622897846346 r, 0.06061532802852698 r, 0.0631303649963022 r,
0.06571208837185505 r, 0.06836123997666599 r, 0.07107855488944881 r,
0.07386476137264342 r, 0.07672058079958999 r, 0.07964672758239233 r,
0.08264390910047736 r, 0.0857128256298576 r, 0.08885417027310427 r,
0.09206862889003742 r, 0.09535688002914089 r, 0.0987195948597075 r,
0.1021574371047232 r, 0.1056710629744951 r, 0.1092611211010309 r,
0.1129282524731764 r, 0.1166730903725168 r, 0.1204962603100498 r,
0.1243983799636342 r, 0.1283800591162231 r, 0.1324418995948859 r,
0.1365844952106265 r, 0.140808431699002 r, 0.1451142866615502 r,
0.1495026295080298 r, 0.1539740213994798 r])
```

Hasilnya sama dengan sebelumnya yang menggunakan persamaan parabola biasa.

Latihan

- Bukalah buku Kalkulus.
- Cari dan pilih beberapa (paling sedikit 5 fungsi berbeda tipe/bentuk/jenis) fungsi dari buku tersebut, kemudian definisikan di EMT pada baris-baris perintah berikut (jika perlu tambahkan lagi).
- Untuk setiap fungsi, tentukan anti turunannya (jika ada), hitunglah integral tentu dengan batas-batas yang menarik (Anda tentukan sendiri), seperti contoh-contoh tersebut.
- Lakukan hal yang sama untuk fungsi-fungsi yang tidak dapat diintegralkan (cari sedikitnya 3 fungsi).
- Gambar grafik fungsi dan daerah integrasinya pada sumbu koordinat yang sama.
- Gunakan integral tentu untuk mencari luas daerah yang dibatasi oleh dua kurva yang berpotongan di dua titik. (Cari dan gambar kedua kurva dan arsir (warnai) daerah yang dibatasi oleh keduanya.)
- Gunakan integral tentu untuk menghitung volume benda putar kurva y= f(x) yang diputar mengelilingi sumbu x dari x=a sampai x=b, yakni

$$V = \int_a^b \pi(f(x)^2\, dx.$$

(Pilih fungsinya dan gambar kurva dan benda putar yang dihasilkan. Anda dapat mencari contoh-contoh bagaimana cara menggambar benda hasil perputaran suatu kurva.)
- Gunakan integral tentu untuk menghitung panjang kurva y=f(x) dari x=a sampai x=b dengan menggunakan rumus:

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

(Pilih fungsi dan gambar kurvanya.)
- Apabila fungsi dinyatakan dalam koordinat kutub x=f(r,t), y=g(r,t), r=h(t), x=a bersesuaian dengan t=t0 dan x=b bersesuian dengan t=t1, maka rumus di atas akan menjadi:

$$S = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

- Pilih beberapa kurva menarik (selain lingkaran dan parabola) dari buku kalkulus. Nyatakan setiap kurva tersebut dalam bentuk:

```
a. koordinat Kartesius (persamaan y=f(x))
b. koordinat kutub ( r=r(theta))
c. persamaan parametrik x=x(t), y=y(t)
d. persamaan implit F(x,y)=0
```

- Tentukan kurvatur masing-masing kurva dengan menggunakan keempat representasi tersebut (hasilnya harus sama).
- Gambarlah kurva asli, kurva kurvatur, kurva jari-jari lingkaran oskulasi, dan salah satu lingkaran oskulasinya.

Integral tak tentu (polinomial dan rasional)

$$integralx^n dx$$

```
>/* antiturunan simbolik */
```

```
"skipcomment" illegal at this place!
Variable antiturunan not found!
Error in:
/* antiturunan simbolik */ ...
              ^
```

```
>$showev('integrate(x^n, x))
```

```
Maxima output too long!
Error in:
$showev('integrate(x^n, x)) ...
                           ^
```

```
>/* bila n symbolic: Maxima menanyakan n; set n dahulu jika perlu */
```

```
"skipcomment" illegal at this place!
Variable bila not found!
Error in:
/* bila n symbolic: Maxima menanyakan n; set n dahulu jika per ...
       ^
```

```
>maxima n:3
```

3

```
>$showev('integrate(x^n, x))
```

```
  Maxima output too long!
  Error in:
  $showev('integrate(x^n, x)) ...
                             ^
```

```
>/* Hitung antiturunan dan integral tentu */
```

```
  "skipcomment" illegal at this place!
  Variable Hitung not found!
  Error in:
  /* Hitung antiturunan dan integral tentu */ ...
            ^
```

```
>maxima f(x) := x^2 * sqrt(2*x + 1)
```

```
          f(x) := [0, 2.7777500001498e-14
                                        2
   sqrt(3.333316666714881e-7 r + 1) r ,
                                                          2
  1.777706668053906e-12 sqrt(2.66661333384044e-6 r + 1) r ,
                                                           2
  2.024817758005038e-11 sqrt(8.999595008676864e-6 r + 1) r ,
                                                           2
  1.137595747549299e-10 sqrt(2.133162673167988e-5 r + 1) r ,
                                                           2
  4.339192840727639e-10 sqrt(4.166145864334392e-5 r + 1) r ,
                                                          2
  1.295533521972174e-9 sqrt(7.198704111080478e-5 r + 1) r ,
                                                          2
  3.266426827094104e-9 sqrt(1.143053249344772e-4 r + 1) r ,
                                                          2
  7.277118895509326e-9 sqrt(1.706120616546125e-4 r + 1) r ,
                                                         2
  1.475029730376073e-8 sqrt(2.42901603977913e-4 r + 1) r ,
                                                          2
  2.775001355397757e-8 sqrt(3.331667063437016e-4 r + 1) r ,
                                                          2
  4.915051879738995e-8 sqrt(4.433983256503793e-4 r + 1) r ,
                                                         2
  8.28246445511412e-8 sqrt(5.755854221612677e-4 r + 1) r ,
                                                         2
  1.33851622723744e-7 sqrt(7.317147606102914e-4 r + 1) r ,
                                                          2
  2.087442283111582e-7 sqrt(9.137707115270399e-4 r + 1) r ,
                                                          2
  3.156951172237287e-7 sqrt(0.001123735052801556 r + 1) r ,
```

```
                                                                  2
4.64842220857938e-7 sqrt(0.001363586771508052 r + 1) r ,
                                                                   2
6.685530482422835e-7 sqrt(0.001635301866007965 r + 1) r ,
                                                                   2
9.417277358666075e-7 sqrt(0.001940853148351629 r + 1) r ,
                                                                  2
1.30212067465563e-6 sqrt(0.002282210046998856 r + 1) r ,
                                                                   2
1.770680532972444e-6 sqrt(0.002661338409877589 r + 1) r ,
                                                                   2
2.371908484044149e-6 sqrt(0.003080200307800873 r + 1) r ,
                                                                   2
3.134234435790633e-6 sqrt(0.003540753838261357 r + 1) r ,
                                                                   2
4.090411050716832e-6 sqrt(0.004044952929623202 r + 1) r ,
                                                                   2
5.277925333300395e-6 sqrt(0.004594747145730826 r + 1) r ,
                                                                   2
6.739427552177104e-6 sqrt(0.005192081490954126 r + 1) r ,
                                                                   2
8.523177254399114e-6 sqrt(0.005838896215689782 r + 1) r ,
                                                                   2
1.068350611911921e-5 sqrt(0.006537126622337741 r + 1) r ,
                                                                   2
1.328129738824626e-5 sqrt(0.007288702871772523 r + 1) r ,
                                                                   2
1.638448160192355e-5 sqrt(0.008095549790328893 r + 1) r ,
                                                                   2
2.006854835710647e-5 sqrt(0.008959586677320885 r + 1) r ,
                                                                  2
2.44170737980647e-5 sqrt(0.009882727113112999 r + 1) r ,
                                                                   2
2.952226353832265e-5 sqrt(0.01086687876776449 r + 1) r ,
                                                                   2
3.548551070434468e-5 sqrt(0.01191394321026329 r + 1) r ,
                                                                   2
4.241796878224187e-5 sqrt(0.01302581571837125 r + 1) r ,
                                                                   2
5.044113893984222e-5 sqrt(0.01420438508909727 r + 1) r ,
                                                                   2
5.968747148772726e-5 sqrt(0.01545153344982009 r + 1) r ,
                                                                   2
7.030098113418114e-5 sqrt(0.01676913607007602 r + 1) r ,
                                                                  2
8.24378756805832e-5 sqrt(0.01815906117403465 r + 1) r ,
                                                                   2
9.626719779540764e-5 sqrt(0.01962316975367717 r + 1) r ,
                                                                 2
1.11971479496896e-4 sqrt(0.021163315382699 r + 1) r ,
                                                                   2
1.297474089664522e-4 sqrt(0.02278134403115428 r + 1) r ,
                                                                   2
1.498065093069853e-4 sqrt(0.02447909388085967 r + 1) r ,
                                                                   2
1.723758288528179e-4 sqrt(0.02625839514157846 r + 1) r ,
```

```
                                                                  2
1.976986426302469e-4 sqrt(0.02812106986800089 r + 1) r ,
                                                                  2
2.260351645605837e-4 sqrt(0.03006893177753966 r + 1) r ,
                                                                  2
2.576632699903951e-4 sqrt(0.03210378606896047 r + 1) r ,
                                                                  2
2.928792281266932e-4 sqrt(0.03422742924186351 r + 1) r ,
                                                                  2
3.319984439480964e-4 sqrt(0.03644164891703428 r + 1) r ,
                                                                  2
3.753562091564763e-4 sqrt(0.03874822365768404 r + 1) r ,
                                                                 2
4.233084617271431e-4 sqrt(0.0411489227915941 r + 1) r ,
                                                                  2
4.762325536095718e-4 sqrt(0.04364550623418506 r + 1) r ,
                                                                  2
5.345280261246171e-4 sqrt(0.04623972431252665 r + 1) r ,
                                                                  2
5.986173925984417e-4 sqrt(0.04893331759030617 r + 1) r ,
                                                                  2
6.689469277678383e-4 sqrt(0.05172801669377391 r + 1) r ,
                                                                  2
7.459874634862211e-4 sqrt(0.05462554213868165 r + 1) r ,
                                                                  2
8.302351902545073e-4 sqrt(0.05762760415823331 r + 1) r ,
                                                                  2
9.222124640960191e-4 sqrt(0.06073590253206151 r + 1) r ,
                                                                  2
0.001022468618290102 sqrt(0.06395212641625303 r + 1) r ,
                                                                  2
0.001131580779474263 sqrt(0.06727795417443261 r + 1) r ,
                                                                  2
0.001250154687620788 sqrt(0.07071505320992943 r + 1) r ,
                                                                  2
0.001378825519389357 sqrt(0.07426507979903763 r + 1) r ,
                                                                  2
0.001518258714353595 sqrt(0.07792967892539004 r + 1) r ,
                                                                2
0.001669150803595751 sqrt(0.081710484115461 r + 1) r ,
                                                                  2
0.001832230240160423 sqrt(0.08560911727521603 r + 1) r ,
                                                                  2
0.002008258230854871 sqrt(0.08962718852792095 r + 1) r ,
                                                                  2
0.002198029568880921 sqrt(0.09376629605313247 r + 1) r ,
                                                                  2
0.002402373466780307 sqrt(0.09802802592688087 r + 1) r ,
                                                                 2
0.002622154389173151 sqrt(0.1024139519630631 r + 1) r ,
                                                                  2
0.002858272884767075 sqrt(0.1069256355560644 r + 1) r ,
                                                                  2
0.003111666417112067 sqrt(0.1115646255246181 r + 1) r ,
                                                                 2
0.003383310193575043 sqrt(0.1163324579569269 r + 1) r ,
```

```
                                                                  2
0.003674217992005929 sqrt(0.121230656057054 r + 1) r ,
                                                                   2
0.003985442984566339 sqrt(0.1262607299926044 r + 1) r ,
                                                                   2
0.004318078558190487 sqrt(0.1314241767437101 r + 1) r ,
                                                                  2
0.004673259131147316 sqrt(0.136722479953332 r + 1) r ,
                                                                   2
0.005052160965172387 sqrt(0.1421571097788976 r + 1) r ,
                                                                   2
0.005456002972637555 sqrt(0.1477295227452868 r + 1) r ,
                                                                 2
0.005886047518226416 sqrt(0.15344116159918 r + 1) r ,
                                                                   2
0.006343601214583815 sqrt(0.1592934551647847 r + 1) r ,
                                                                   2
0.006830015711407966 sqrt(0.1652878182009547 r + 1) r ,
                                                                   2
0.007346688477454374 sqrt(0.1714256512597152 r + 1) r ,
                                                                   2
0.007895063574921807 sqrt(0.1777083405462085 r + 1) r ,
                                                                   2
0.008476632425691433 sqrt(0.1841372577800748 r + 1) r ,
                                                                   2
0.009092934568891969 sqrt(0.1907137600582818 r + 1) r ,
                                                                   2
0.009745558409264787 sqrt(0.197439189719415 r + 1) r ,
                                                                   2
0.01043614195580549 sqrt(0.2043148742094465 r + 1) r ,
                                                                   2
0.01116637355015972 sqrt(0.2113421259489903 r + 1) r ,
                                                                   2
0.01193799258425414 sqrt(0.2185222422020618 r + 1) r ,
                                                                   2
0.01275279020664547 sqrt(0.2258565049463528 r + 1) r ,
                                                                   2
0.01361261001707348 sqrt(0.2333461807450337 r + 1) r ,
                                                                   2
0.01451934874870728 sqrt(0.2409925206200996 r + 1) r ,
                                                                   2
0.01547495693757671 sqrt(0.2487967599272685 r + 1) r ,
                                                                   2
0.01648143957868493 sqrt(0.2567601182324462 r + 1) r ,
                                                                   2
0.01754085676830185 sqrt(0.2648837991897719 r + 1) r ,
                                                                   2
0.01865532433194167 sqrt(0.2731689904212531 r + 1) r ,
                                                                   2
0.01982701443753252 sqrt(0.2816168633980041 r + 1) r ,
                                                                   2
0.02105815619329058 sqrt(0.2902285733231005 r + 1) r ,
                                                                   2
0.02235103622981523 sqrt(0.2990052590160597 r + 1) r ,
                                                                   2
0.02370799926592746 sqrt(0.3079480427989596 r + 1) r ]
```

```
>$showev('integrate(f(x), x))
```

```
  Maxima output too long!
  Error in:
  $showev('integrate(f(x), x)) ...
                              ^
```

```
>$showev('integrate(f(x), x, 0, 2))
```

```
  Maxima said:
  defint: variable of integration must be a simple or subscripted variable.
  defint: found errexp1
  #0: showev(f='integrate([0,2.7777500001498e-14*sqrt(3.333316666714881e-7*r+1)*r^2,1.7777
   -- an error. To debug this try: debugmode(true);

  Error in:
  $showev('integrate(f(x), x, 0, 2)) ...
                                    ^
```

Mohon maaf latihan awal sub bab ini gagal :)

Coba lagi

Aplikasi (luas daerah)

```
>/* luas daerah antara y = x^3 - x dan y = -x^2 pada dua titik potong di [0,(sqrt(5)-1)/2]
```

```
  "skipcomment" illegal at this place!
  Variable luas not found!
  Error in:
  /* luas daerah antara y = x^3 - x dan y = -x^2 pada dua titik  ...
          ^
```

```
>maxima a:0
```

$$0$$

```
>maxima b:(sqrt(5)-1)/2
```

$$\frac{-1+\sqrt{5}}{2}$$

```
>maxima $showev('integrate(-x^2 - (x^3 - x), x, a, b))
```

```
  defint: variable of integration must be a simple or subscripted variable.
  defint: found errexp1
  #0: showev(f='integrate([0,1.66665833335744e-7*r-2.7777500001498e-14*r^2-4.6295601857332
   -- an error. To debug this try: debugmode(true);
```

```
>maxima float(%)
```

```
                                    true
```

Jawaban:

$$\tilde{}0.07581917135421037.$$

Integral tak elementer (numeric)

```
>/* Integral xx dari 0 sampai 1 (tidak punya antiturunan elementer) */
```

```
  "skipcomment" illegal at this place!
  Variable Integral not found!
  Error in:
  /* Integral xx dari 0 sampai 1 (tidak punya antiturunan elemen ...
              ^
```

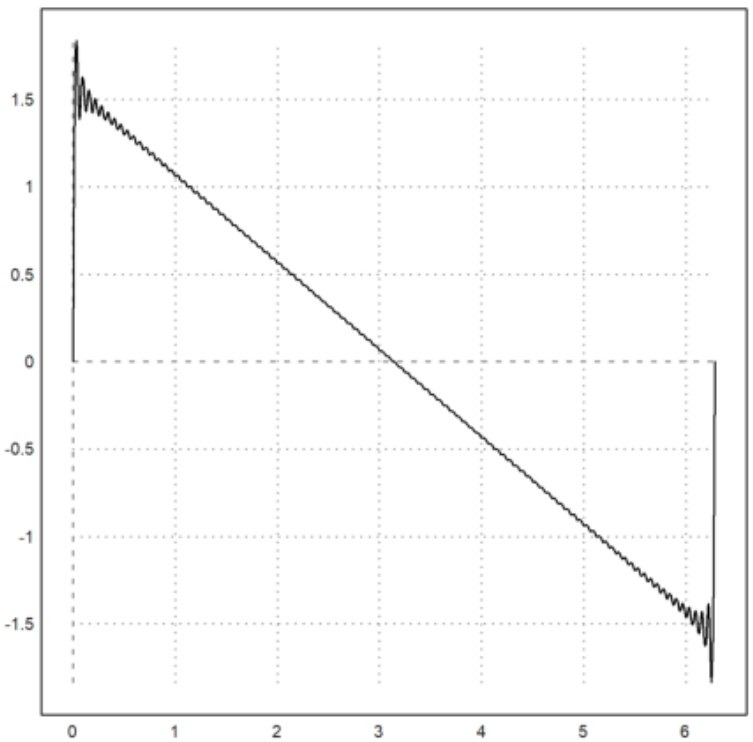
```
>maxima f(x) := x^x
```

```
          f(x) := expt([0, 1.66665833335744e-7 r,
  1.33330666692022e-6 r, 4.499797504338432e-6 r,
  1.066581336583994e-5 r, 2.083072932167196e-5 r,
  3.599352055540239e-5 r, 5.71526624672386e-5 r,
  8.530603082730626e-5 r, 1.214508019889565e-4 r,
  1.665833531718508e-4 r, 2.216991628251896e-4 r,
  2.877927110806339e-4 r, 3.658573803051457e-4 r, 4.5688535576352e-4 r,
  5.618675264007778e-4 r, 6.817933857540259e-4 r,
  8.176509330039827e-4 r, 9.704265741758145e-4 r,
  0.001141105023499428 r, 0.001330669204938795 r,
  0.001540100153900437 r, 0.001770376919130678 r,
  0.002022476464811601 r, 0.002297373572865413 r,
  0.002596040745477063 r, 0.002919448107844891 r,
  0.003268563311168871 r, 0.003644351435886262 r,
  0.004047774895164447 r, 0.004479793338660443 r, 0.0049413635565565 r,
  0.005433439383882244 r, 0.005956971605131645 r,
  0.006512907859185624 r, 0.007102192544548636 r,
```

0.007725766724910044 r, 0.00838456803503801 r,
0.009079530587017326 r, 0.009811584876838586 r, 0.0105816576913495 r,
0.01139067201557714 r, 0.01223954694042984 r, 0.01312919757078923 r,
0.01406053493400045 r, 0.01503446588876983 r, 0.01605189303448024 r,
0.01711371462093175 r, 0.01822082445851714 r, 0.01937411182884202 r,
0.02057446139579705 r, 0.02182275311709253 r, 0.02311986215626333 r,
0.02446665879515308 r, 0.02586400834688696 r, 0.02731277106934082 r,
0.02881380207911666 r, 0.03036795126603076 r, 0.03197606320812652 r,
0.0336389770872163 r, 0.03535752660496472 r, 0.03713253989951881 r,
0.03896483946269502 r, 0.0408552420577305 r, 0.04280455863760801 r,
0.04481359426396048 r, 0.04688314802656623 r, 0.04901401296344043 r,
0.05120697598153157 r, 0.05346281777803219 r, 0.05578231276230905 r,
0.05816622897846346 r, 0.06061532802852698 r, 0.0631303649963022 r,
0.06571208837185505 r, 0.06836123997666599 r, 0.07107855488944881 r,
0.07386476137264342 r, 0.07672058079958999 r, 0.07964672758239233 r,
0.08264390910047736 r, 0.0857128256298576 r, 0.08885417027310427 r,
0.09206862889003742 r, 0.09535688002914089 r, 0.0987195948597075 r,
0.1021574371047232 r, 0.1056710629744951 r, 0.1092611211010309 r,
0.1129282524731764 r, 0.1166730903725168 r, 0.1204962603100498 r,
0.1243983799636342 r, 0.1283800591162231 r, 0.1324418995948859 r,
0.1365844952106265 r, 0.140808431699002 r, 0.1451142866615502 r,
0.1495026295080298 r, 0.1539740213994798 r],
[0, 1.66665833335744e-7 r, 1.33330666692022e-6 r,
4.499797504338432e-6 r, 1.066581336583994e-5 r,
2.083072932167196e-5 r, 3.599352055540239e-5 r,
5.71526624672386e-5 r, 8.530603082730626e-5 r,
1.214508019889565e-4 r, 1.665833531718508e-4 r,
2.216991628251896e-4 r, 2.877927110806339e-4 r,
3.658573803051457e-4 r, 4.5688535576352e-4 r, 5.618675264007778e-4 r,
6.817933857540259e-4 r, 8.176509330039827e-4 r,
9.704265741758145e-4 r, 0.001141105023499428 r,
0.001330669204938795 r, 0.001540100153900437 r,
0.001770376919130678 r, 0.002022476464811601 r,
0.002297373572865413 r, 0.002596040745477063 r,
0.002919448107844891 r, 0.003268563311168871 r,
0.003644351435886262 r, 0.004047774895164447 r,
0.004479793338660443 r, 0.0049413635565565 r, 0.005433439383882244 r,
0.005956971605131645 r, 0.006512907859185624 r,
0.007102192544548636 r, 0.007725766724910044 r,
0.00838456803503801 r, 0.009079530587017326 r,
0.009811584876838586 r, 0.0105816576913495 r, 0.01139067201557714 r,
0.01223954694042984 r, 0.01312919757078923 r, 0.01406053493400045 r,
0.01503446588876983 r, 0.01605189303448024 r, 0.01711371462093175 r,
0.01822082445851714 r, 0.01937411182884202 r, 0.02057446139579705 r,
0.02182275311709253 r, 0.02311986215626333 r, 0.02446665879515308 r,
0.02586400834688696 r, 0.02731277106934082 r, 0.02881380207911666 r,
0.03036795126603076 r, 0.03197606320812652 r, 0.0336389770872163 r,
0.03535752660496472 r, 0.03713253989951881 r, 0.03896483946269502 r,
0.0408552420577305 r, 0.04280455863760801 r, 0.04481359426396048 r,
0.04688314802656623 r, 0.04901401296344043 r, 0.05120697598153157 r,
0.05346281777803219 r, 0.05578231276230905 r, 0.05816622897846346 r,
0.06061532802852698 r, 0.0631303649963022 r, 0.06571208837185505 r,
0.06836123997666599 r, 0.07107855488944881 r, 0.07386476137264342 r,
0.07672058079958999 r, 0.07964672758239233 r, 0.08264390910047736 r,
0.0857128256298576 r, 0.08885417027310427 r, 0.09206862889003742 r,
0.09535688002914089 r, 0.0987195948597075 r, 0.1021574371047232 r,

```
0.1056710629744951 r, 0.1092611211010309 r, 0.1129282524731764 r,
0.1166730903725168 r, 0.1204962603100498 r, 0.1243983799636342 r,
0.1283800591162231 r, 0.1324418995948859 r, 0.1365844952106265 r,
0.140808431699002 r, 0.1451142866615502 r, 0.1495026295080298 r,
0.1539740213994798 r])
```

```
>integrate("x^x", 0, 1): /* numeric approx */
```




```
"skipcomment" illegal at this place!
Variable numeric not found!
Error in:
integrate("x^x", 0, 1): /* numeric approx */ ...
                                    ^
```

Jawaban singkat:

```
~ 0.7834935879025506 (hasil aproksimasi numerik).
```

Antiturunan & definite

```
>/* Hitung antiturunan dan integral tentu */
```

```
"skipcomment" illegal at this place!
Variable Hitung not found!
Error in:
/* Hitung antiturunan dan integral tentu */ ...
          ^
```

```
>maxima f(x) := x^2 * sqrt(2*x + 1)
```

```
         f(x) := [0, 2.7777500001498e-14
                                        2
  sqrt(1 + 3.333316666714881e-7 r) r ,
                                                          2
 1.777706668053906e-12 sqrt(1 + 2.66661333384044e-6 r) r ,
                                                           2
 2.024817758005038e-11 sqrt(1 + 8.999595008676864e-6 r) r ,
                                                           2
 1.137595747549299e-10 sqrt(1 + 2.133162673167988e-5 r) r ,
                                                           2
 4.339192840727639e-10 sqrt(1 + 4.166145864334392e-5 r) r ,
                                                          2
 1.295533521972174e-9 sqrt(1 + 7.198704111080478e-5 r) r ,
                                                          2
 3.266426827094104e-9 sqrt(1 + 1.143053249344772e-4 r) r ,
                                                          2
 7.277118895509326e-9 sqrt(1 + 1.706120616546125e-4 r) r ,
                                                         2
 1.475029730376073e-8 sqrt(1 + 2.42901603977913e-4 r) r ,
                                                          2
 2.775001355397757e-8 sqrt(1 + 3.331667063437016e-4 r) r ,
                                                          2
 4.915051879738995e-8 sqrt(1 + 4.433983256503793e-4 r) r ,
                                                         2
 8.28246445511412e-8 sqrt(1 + 5.755854221612677e-4 r) r ,
                                                         2
 1.33851622723744e-7 sqrt(1 + 7.317147606102914e-4 r) r ,
                                                          2
 2.087442283111582e-7 sqrt(1 + 9.137707115270399e-4 r) r ,
                                                          2
 3.156951172237287e-7 sqrt(1 + 0.001123735052801556 r) r ,
                                                         2
 4.64842220857938e-7 sqrt(1 + 0.001363586771508052 r) r ,
                                                          2
 6.685530482422835e-7 sqrt(1 + 0.001635301866007965 r) r ,
                                                          2
 9.417277358666075e-7 sqrt(1 + 0.001940853148351629 r) r ,
                                                         2
 1.30212067465563e-6 sqrt(1 + 0.002282210046998856 r) r ,
                                                          2
 1.770680532972444e-6 sqrt(1 + 0.002661338409877589 r) r ,
                                                          2
 2.371908484044149e-6 sqrt(1 + 0.003080200307800873 r) r ,
                                                          2
 3.134234435790633e-6 sqrt(1 + 0.003540753838261357 r) r ,
                                                          2
 4.090411050716832e-6 sqrt(1 + 0.004044952929623202 r) r ,
                                                          2
 5.277925333300395e-6 sqrt(1 + 0.004594747145730826 r) r ,
                                                          2
 6.739427552177104e-6 sqrt(1 + 0.005192081490954126 r) r ,
                                                          2
```

```
8.523177254399114e-6 sqrt(1 + 0.005838896215689782 r) r ,
                                                        2
1.068350611911921e-5 sqrt(1 + 0.00653712662233774 1 r) r ,
                                                         2
1.328129738824626e-5 sqrt(1 + 0.007288702871772523 r) r ,
                                                         2
1.638448160192355e-5 sqrt(1 + 0.008095549790328893 r) r ,
                                                         2
2.006854835710647e-5 sqrt(1 + 0.008959586677320885 r) r ,
                                                        2
2.44170737980647e-5 sqrt(1 + 0.009882727113112999 r) r ,
                                                        2
2.952226353832265e-5 sqrt(1 + 0.01086687876776449 r) r ,
                                                        2
3.548551070434468e-5 sqrt(1 + 0.01191394321026329 r) r ,
                                                        2
4.241796878224187e-5 sqrt(1 + 0.01302581571837125 r) r ,
                                                        2
5.044113893984222e-5 sqrt(1 + 0.01420438508909727 r) r ,
                                                        2
5.968747148772726e-5 sqrt(1 + 0.01545153344982009 r) r ,
                                                        2
7.030098113418114e-5 sqrt(1 + 0.01676913607007602 r) r ,
                                                       2
8.24378756805832e-5 sqrt(1 + 0.01815906117403465 r) r ,
                                                        2
9.626719779540764e-5 sqrt(1 + 0.01962316975367717 r) r ,
                                                     2
1.11971479496896e-4 sqrt(1 + 0.021163315382699 r) r ,
                                                        2
1.297474089664522e-4 sqrt(1 + 0.02278134403115428 r) r ,
                                                        2
1.498065093069853e-4 sqrt(1 + 0.02447909388085967 r) r ,
                                                        2
1.723758288528179e-4 sqrt(1 + 0.02625839514157846 r) r ,
                                                        2
1.976986426302469e-4 sqrt(1 + 0.02812106986800089 r) r ,
                                                        2
2.260351645605837e-4 sqrt(1 + 0.03006893177753966 r) r ,
                                                        2
2.576632699903951e-4 sqrt(1 + 0.03210378606896047 r) r ,
                                                        2
2.928792281266932e-4 sqrt(1 + 0.03422742924186351 r) r ,
                                                        2
3.319984439480964e-4 sqrt(1 + 0.03644164891703428 r) r ,
                                                        2
3.753562091564763e-4 sqrt(1 + 0.03874822365768404 r) r ,
                                                       2
4.233084617271431e-4 sqrt(1 + 0.0411489227915941 r) r ,
                                                        2
4.762325536095718e-4 sqrt(1 + 0.04364550623418506 r) r ,
                                                        2
5.345280261246171e-4 sqrt(1 + 0.04623972431252665 r) r ,
                                                        2
5.986173925984417e-4 sqrt(1 + 0.04893331759030617 r) r ,
                                                        2
```

```
6.689469277678383e-4 sqrt(1 + 0.05172801669377391 r) r ,
                                                        2
7.459874634862211e-4 sqrt(1 + 0.05462554213868165 r) r ,
                                                        2
8.302351902545073e-4 sqrt(1 + 0.05762760415823331 r) r ,
                                                        2
9.222124640960191e-4 sqrt(1 + 0.06073590253206151 r) r ,
                                                        2
0.001022468618290102 sqrt(1 + 0.06395212641625303 r) r ,
                                                        2
0.001131580779474263 sqrt(1 + 0.06727795417443261 r) r ,
                                                        2
0.001250154687620788 sqrt(1 + 0.07071505320992943 r) r ,
                                                        2
0.001378825519389357 sqrt(1 + 0.07426507979903763 r) r ,
                                                        2
0.001518258714353595 sqrt(1 + 0.07792967892539004 r) r ,
                                                       2
0.001669150803595751 sqrt(1 + 0.081710484115461 r) r ,
                                                        2
0.001832230240160423 sqrt(1 + 0.08560911727521603 r) r ,
                                                        2
0.002008258230854871 sqrt(1 + 0.08962718852792095 r) r ,
                                                        2
0.002198029568880921 sqrt(1 + 0.09376629605313247 r) r ,
                                                        2
0.002402373466780307 sqrt(1 + 0.09802802592688087 r) r ,
                                                        2
0.002622154389173151 sqrt(1 + 0.1024139519630631 r) r ,
                                                        2
0.002858272884767075 sqrt(1 + 0.1069256355560644 r) r ,
                                                        2
0.003111666417112067 sqrt(1 + 0.1115646255246181 r) r ,
                                                        2
0.003383310193575043 sqrt(1 + 0.1163324579569269 r) r ,
                                                       2
0.003674217992005929 sqrt(1 + 0.121230656057054 r) r ,
                                                        2
0.003985442984566339 sqrt(1 + 0.1262607299926044 r) r ,
                                                        2
0.004318078558190487 sqrt(1 + 0.1314241767437101 r) r ,
                                                       2
0.004673259131147316 sqrt(1 + 0.136722479953332 r) r ,
                                                        2
0.005052160965172387 sqrt(1 + 0.1421571097788976 r) r ,
                                                        2
0.005456002972637555 sqrt(1 + 0.1477295227452868 r) r ,
                                                      2
0.005886047518226416 sqrt(1 + 0.15344116159918 r) r ,
                                                        2
0.006343601214583815 sqrt(1 + 0.1592934551647847 r) r ,
                                                        2
0.006830015711407966 sqrt(1 + 0.1652878182009547 r) r ,
                                                        2
0.007346688477454374 sqrt(1 + 0.1714256512597152 r) r ,
                                                        2
```

```
0.007895063574921807 sqrt(1 + 0.1777083405462085 r) r ,
                                                      2
0.008476632425691433 sqrt(1 + 0.1841372577800748 r) r ,
                                                      2
0.009092934568891969 sqrt(1 + 0.1907137600582818 r) r ,
                                                     2
0.009745558409264787 sqrt(1 + 0.197439189719415 r) r ,
                                                     2
0.01043614195580549 sqrt(1 + 0.2043148742094465 r) r ,
                                                     2
0.01116637355015972 sqrt(1 + 0.2113421259489903 r) r ,
                                                     2
0.01193799258425414 sqrt(1 + 0.2185222422020618 r) r ,
                                                     2
0.01275279020664547 sqrt(1 + 0.2258565049463528 r) r ,
                                                     2
0.01361261001707348 sqrt(1 + 0.2333461807450337 r) r ,
                                                     2
0.01451934874870728 sqrt(1 + 0.2409925206200996 r) r ,
                                                     2
0.01547495693757671 sqrt(1 + 0.2487967599272685 r) r ,
                                                     2
0.01648143957868493 sqrt(1 + 0.2567601182324462 r) r ,
                                                     2
0.01754085676830185 sqrt(1 + 0.2648837991897719 r) r ,
                                                     2
0.01865532433194167 sqrt(1 + 0.2731689904212531 r) r ,
                                                     2
0.01982701443753252 sqrt(1 + 0.2816168633980041 r) r ,
                                                     2
0.02105815619329058 sqrt(1 + 0.2902285733231005 r) r ,
                                                     2
0.02235103622981523 sqrt(1 + 0.2990052590160597 r) r ,
                                                     2
0.02370799926592746 sqrt(1 + 0.3079480427989596 r) r ]
```

```
>$showev('integrate(f(x), x))
```

```
  Maxima output too long!
  Error in:
  $showev('integrate(f(x), x)) ...
                              ^
```

```
>maxima $showev('integrate(f(x), x, 0, 2))
```

```
  defint: variable of integration must be a simple or subscripted variable.
  defint: found errexp1
  #0: showev(f='integrate([0,2.7777500001498e-14*sqrt(1+3.333316666714881e-7*r)*r^2,1.7777
   -- an error. To debug this try: debugmode(true);
```

Jawaban singkat: Maxima mengembalikan bentuk eksak; nilai tertentu adalah 25/7*sqrt(2) - 2/105*2 (lihat file untuk bentuk disederhanakan) — jalankan float(...) untuk nilai decimal.

Barisan dan Deret

(Catatan: bagian ini belum lengkap. Anda dapat membaca contoh-contoh pengguanaan EMT dan Maxima untuk menghitung limit barisan, rumus jumlah parsial suatu deret, jumlah tak hingga suatu deret konvergen, dan sebagainya. Anda dapat mengeksplor contoh-contoh di EMT atau perbagai panduan penggunaan Maxima di software Maxima atau dari Internet.)

Barisan dapat didefinisikan dengan beberapa cara di dalam EMT, di antaranya:

- dengan cara yang sama seperti mendefinisikan vektor dengan elemen-elemen beraturan (menggunakan titik dua ":");
- menggunakan perintah "sequence" dan rumus barisan (suku ke -n);
- menggunakan perintah "iterate" atau "niterate";
- menggunakan fungsi Maxima "create_list" atau "makelist" untuk menghasilkan barisan simbolik;
- menggunakan fungsi biasa yang inputnya vektor atau barisan;
- menggunakan fungsi rekursif.

EMT menyediakan beberapa perintah (fungsi) terkait barisan, yakni:

- sum: menghitung jumlah semua elemen suatu barisan
- cumsum: jumlah kumulatif suatu barisan
- differences: selisih antar elemen-elemen berturutan

EMT juga dapat digunakan untuk menghitung jumlah deret berhingga maupun deret tak hingga, dengan menggunakan perintah (fungsi) "sum". Perhitungan dapat dilakukan secara numerik maupun simbolik dan eksak.

Berikut adalah beberapa contoh perhitungan barisan dan deret menggunakan EMT.

```
>1:10 // barisan sederhana
```

```
  [1,  2,  3,  4,  5,  6,  7,  8,  9,  10]
```

```
>1:2:30
```

```
  [1,  3,  5,  7,  9,  11,  13,  15,  17,  19,  21,  23,  25,  27,  29]
```

Iterasi dan Barisan

EMT menyediakan fungsi iterate("g(x)", x0, n) untuk melakukan iterasi

$$x_{k+1} = g(x_k),\ x_0 = x_0, k = 1, 2, 3, ..., n.$$

Berikut ini disajikan contoh-contoh penggunaan iterasi dan rekursi dengan EMT. Contoh pertama menunjukkan pertumbuhan dari nilai awal 1000 dengan laju pertambahan 5%, selama 10 periode.

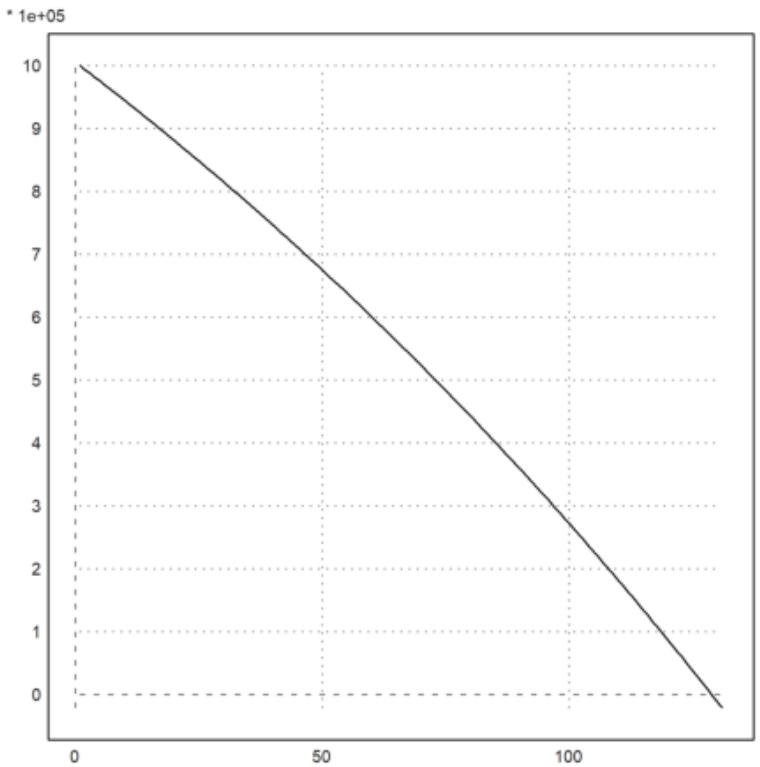
```
>q=1.05; iterate("x*q",1000,n=10)'
```

```
          1000
          1050
        1102.5
       1157.63
       1215.51
       1276.28
```

```
 1340.1
 1407.1
1477.46
1551.33
1628.89
```

Contoh berikutnya memperlihatkan bahaya menabung di bank pada masa sekarang! Dengan bunga tabungan sebesar 6% per tahun atau 0.5% per bulan dipotong pajak 20%, dan biaya administrasi 10000 per bulan, tabungan sebesar 1 juta tanpa diambil selama sekitar 10 tahunan akan habis diambil oleh bank!

```
>r=0.005; plot2d(iterate("(1+0.8*r)*x-10000",1000000,n=130)):
```



Silakan Anda coba-coba, dengan tabungan minimal berapa agar tidak akan habis diambil oleh bank dengan ketentuan bunga dan biaya administrasi seperti di atas.

Berikut adalah perhitungan minimal tabungan agar aman di bank dengan bunga sebesar r dan biaya administrasi a, pajak bunga 20%.

```
>$solve(0.8*r*A-a,A), $% with [r=0.005, a=10]
```

$$\left[A = \frac{5\,a}{4\,r}\right]$$

$$[A = 2500.0]$$

Berikut didefinisikan fungsi untuk menghitung saldo tabungan, kemudian dilakukan iterasi.

```
>maxima function saldo(x,r,a) := round((1+0.8*r)*x-a,2)
```

```
saldo(x, r, a) := 'round((1 + 0.8 r) x - a, 2)
```

```
>iterate({{"saldo",0.005,10}},1000,n=6)
```

```
  [1000,  994,  987.98,  981.93,  975.86,  969.76,  963.64]
```

```
>iterate({{"saldo",0.005,10}},2000,n=6)
```

```
  [2000,  1998,  1995.99,  1993.97,  1991.95,  1989.92,  1987.88]
```

```
>iterate({{"saldo",0.005,10}},2500,n=6)
```

```
  [2500,  2500,  2500,  2500,  2500,  2500,  2500]
```

Tabungan senilai 2,5 juta akan aman dan tidak akan berubah nilai (jika tidak ada penarikan), sedangkan jika tabungan awal kurang dari 2,5 juta, lama kelamaan akan berkurang meskipun tidak pernah dilakukan penarikan uang tabungan.

```
>iterate({{"saldo",0.005,10}},3000,n=6)
```

```
  [3000,  3002,  3004.01,  3006.03,  3008.05,  3010.08,  3012.12]
```

Tabungan yang lebih dari 2,5 juta baru akan bertambah jika tidak ada penarikan.

Untuk barisan yang lebih kompleks dapat digunakan fungsi "sequence()". Fungsi ini menghitung nilai-nilai x[n] dari semua nilai sebelumnya, x[1],...,x[n-1] yang diketahui.
Berikut adalah contoh barisan Fibonacci.

$$x_n = x_{n-1} + x_{n-2}, \quad x_1 = 1, \quad x_2 = 1$$

```
>sequence("x[n-1]+x[n-2]",[1,1],15)
```

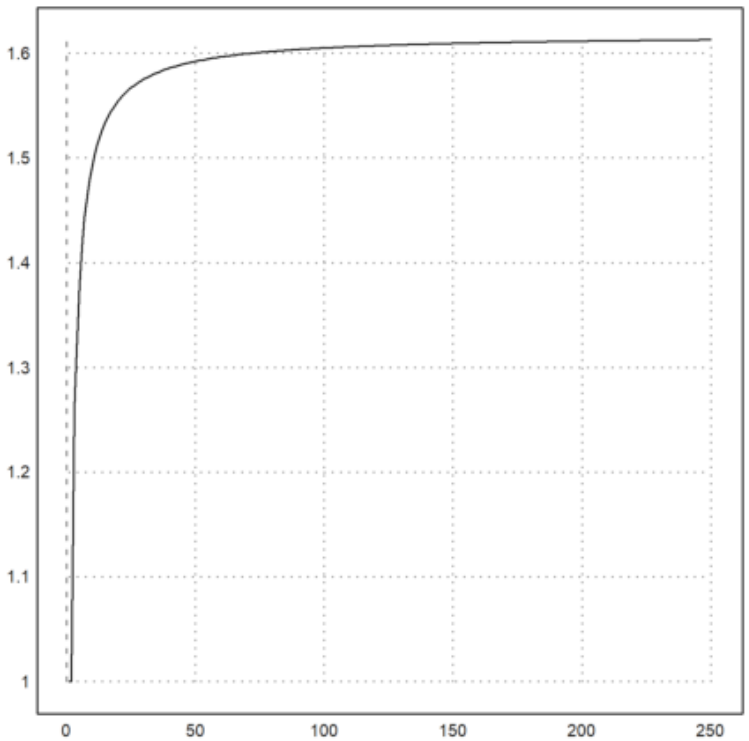
```
  [1,  1,  2,  3,  5,  8,  13,  21,  34,  55,  89,  144,  233,  377,  610]
```

Barisan Fibonacci memiliki banyak sifat menarik, salah satunya adalah akar pangkat ke-n suku ke-n akan konvergen ke pecahan emas:

```
>$'(1+sqrt(5))/2=float((1+sqrt(5))/2)
```

$$\frac{\sqrt{5}+1}{2} = 1.618033988749895$$

```
>plot2d(sequence("x[n-1]+x[n-2]",[1,1],250)^(1/(1:250))):
```



Barisan yang sama juga dapat dihasilkan dengan menggunakan loop.

```
>x=ones(500); for k=3 to 500; x[k]=x[k-1]+x[k-2]; end;
```

Rekursi dapat dilakukan dengan menggunakan rumus yang tergantung pada semua elemen sebelumnya. Pada contoh berikut, elemen ke-n merupakan jumlah (n-1) elemen sebelumnya, dimulai dengan 1 (elemen ke-1). Jelas, nilai elemen ke-n adalah 2^(n-2), untuk n=2, 4, 5, ...

```
>sequence("sum(x)",1,10)
```

```
  [1,  1,  2,  4,  8,  16,  32,  64,  128,  256]
```

Selain menggunakan ekspresi dalam x dan n, kita juga dapat menggunakan fungsi.
Pada contoh berikut, digunakan iterasi

$$x_n = A \cdot x_{n-1},$$

dengan A suatu matriks 2x2, dan setiap x[n] merupakan matriks/vektor 2x1.

```
>A=[1,1;1,2]; function suku(x,n) := A.x[,n-1]
>sequence("suku",[1;1],6)
```

```
  Real 2 x 6 matrix

            1             2             5            13      ...
            1             3             8            21      ...
```

Hasil yang sama juga dapat diperoleh dengan menggunakan fungsi perpangkatan matriks "matrixpower()". Cara ini lebih cepat, karena hanya menggunakan perkalian matriks sebanyak log_2(n).

$$x_n = A.x_{n-1} = A^2.x_{n-2} = A^3.x_{n-3} = ... = A^{n-1}.x_1.$$

```
>sequence("matrixpower(A,n).[1;1]",1,6)
```

```
  Real 2 x 6 matrix

            1             5            13            34      ...
            1             8            21            55      ...
```

Spiral Theodorus

image: Spiral_of_Theodorus.png
Spiral Theodorus (spiral segitiga siku-siku) dapat digambar secara rekursif. Rumus rekursifnya adalah:

$$x_n = \left(1 + \frac{i}{\sqrt{n-1}}\right) x_{n-1}, \quad x_1 = 1,$$

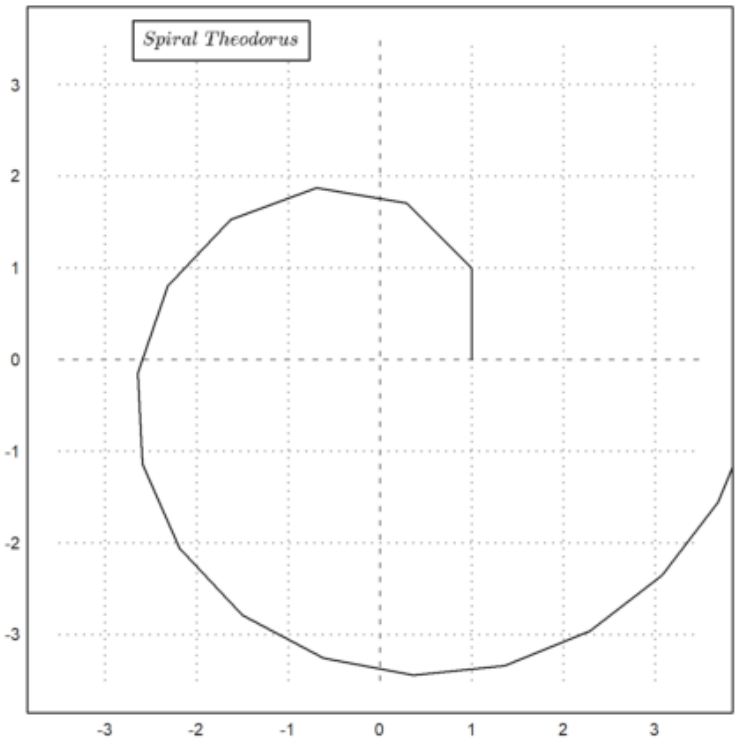
yang menghasilkan barisan bilangan kompleks.

```
>maxima function g(n) := 1+I/sqrt(n)
```

```
                    I
g(n) := 1 + -------
                sqrt(n)
```

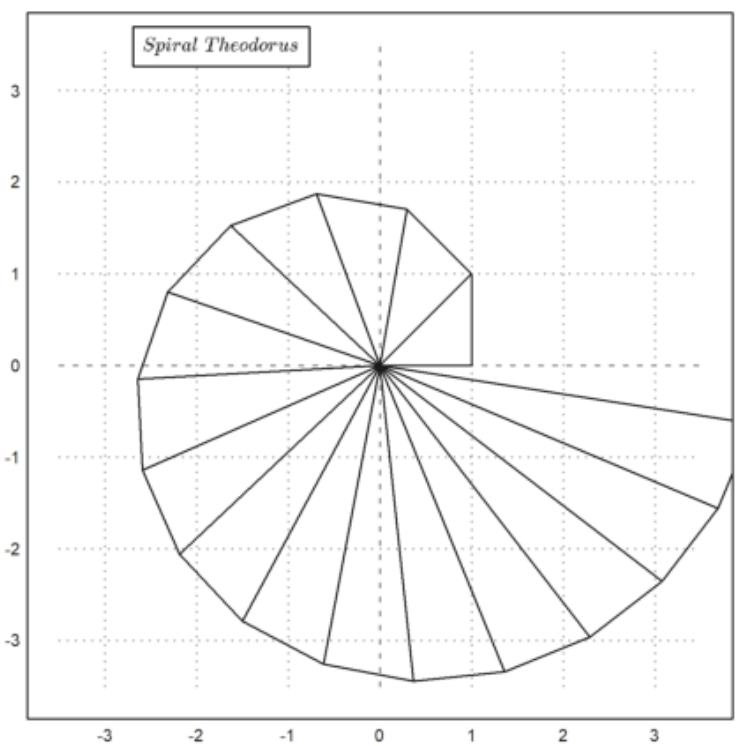
Rekursinya dapat dijalankan sebanyak 17 untuk menghasilkan barisan 17 bilangan kompleks, kemudian digambar bilangan-bilangan kompleksnya.

```
>x=sequence("g(n-1)*x[n-1]",1,17); plot2d(x,r=3.5); textbox(latex("Spiral\ Theodorus"),0.4
```



Selanjutnya dihubungan titik 0 dengan titik-titik kompleks tersebut menggunakan loop.

```
>for i=1:cols(x); plot2d([0,x[i]],>add); end:
```



```
>
```

Spiral tersebut juga dapat didefinisikan menggunakan fungsi rekursif, yang tidak memmerlukan indeks dan bilangan kompleks. Dalam hal ini diigunakan vektor kolom pada bidang.

```
>function gstep (v) ...
```

```
  endfunction
  w=[-v[2];v[1]];
  return v+w/norm(w);
  endfunction
```

Jika dilakukan iterasi 16 kali dimulai dari [1;0] akan didapatkan matriks yang memuat vektor-vektor dari setiap iterasi.

```
>x=iterate("gstep",[1;0],16); plot2d(x[1],x[2],r=3.5,>points):
```

```
  A function returned no value. Cannot assign this to a variable.
  niterate:
      x=f$(x,args());
  Try "trace errors" to inspect local variables after errors.
  iterate:
      return niterate(f$,x,n,till;args());
```

Kekonvergenan

Terkadang kita ingin melakukan iterasi sampai konvergen. Apabila iterasinya tidak konvergen setelah ditunggu lama, Anda dapat menghentikannya dengan menekan tombol [ESC].

```
>iterate("cos(x)",1) // iterasi x(n+1)=cos(x(n)), dengan x(0)=1.
```

```
  0.739085133216
```

Iterasi tersebut konvergen ke penyelesaian persamaan

$$x = \cos(x).$$

Iterasi ini juga dapat dilakukan pada interval, hasilnya adalah barisan interval yang memuat akar tersebut.

```
>hasil := iterate("cos(x)",~1,2~) //iterasi x(n+1)=cos(x(n)), dengan interval awal (1, 2)
```

```
  ~0.739085133211,0.7390851332133~
```

Jika interval hasil tersebut sedikit diperlebar, akan terlihat bahwa interval tersebut memuat akar persamaan x=cos(x).

```
>h=expand(hasil,100), cos(h) << h
```

```
  ~0.73908513309,0.73908513333~
  1
```

Iterasi juga dapat digunakan pada fungsi yang didefinisikan.

```
>function f(x) := (x+2/x)/2
```

Iterasi x(n+1)=f(x(n)) akan konvergen ke akar kuadrat 2.

```
>iterate("f",2), sqrt(2)
```

```
  1.41421356237
  1.41421356237
```

Jika pada perintah iterate diberikan tambahan parameter n, maka hasil iterasinya akan ditampilkan mulai dari iterasi pertama sampai ke-n.

```
>iterate("f",2,5)
```

```
  [2,  1.5,  1.41667,  1.41422,  1.41421,  1.41421]
```

Untuk iterasi ini tidak dapat dilakukan terhadap interval.

```
>niterate("f",~1,2~,5)
```

```
  [ ~1,2~,  ~1,2~,  ~1,2~,  ~1,2~,  ~1,2~,  ~1,2~ ]
```

Perhatikan, hasil iterasinya sama dengan interval awal. Alasannya adalah perhitungan dengan interval bersifat terlalu longgar. Untuk meingkatkan perhitungan pada ekspresi dapat digunakan pembagian intervalnya, menggunakan fungsi ieval().

```
>function s(x) := ieval("(x+2/x)/2",x,10)
```

```
  incorrect syntax: > is not a prefix operator
  d(x[1],x[2],r=3.5,>p
                    ^
  --->(%i610) <---
                    s(x) := ieval("(x+2/x)/2", x, 10)
```

Selanjutnya dapat dilakukan iterasi hingga diperoleh hasil optimal, dan intervalnya tidak semakin mengecil. Hasilnya berupa interval yang memuat akar persamaan:

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right).$$

Satu-satunya solusi adalah

$$x = \sqrt{2}.$$

```
>iterate("s",~1,2~)
```

```
  ~1.41421356236,1.41421356239~
```

Fungsi "iterate()" juga dapat bekerja pada vektor. Berikut adalah contoh fungsi vektor, yang menghasilkan rata-rata aritmetika dan rata-rata geometri.

$$(a_{n+1}, b_{n+1}) = \left(\frac{a_n + b_n}{2}, \sqrt{a_n b_n}\right)$$

Iterasi ke-n disimpan pada vektor kolom x[n].

```
>function g(x) := [(x[1]+x[2])/2;sqrt(x[1]*x[2])]
```

Iterasi dengan menggunakan fungsi tersebut akan konvergen ke rata-rata aritmetika dan geometri dari nilai-nilai awal.

```
>iterate("g",[1;5])
```

```
        2.60401
        2.60401
```

Hasil tersebut konvergen agak cepat, seperti kita cek sebagai berikut.

```
>iterate("g",[1;5],4)
```

```
             1             3       2.61803       2.60403       2.60401
             5       2.23607       2.59002       2.60399       2.60401
```

Iterasi pada interval dapat dilakukan dan stabil, namun tidak menunjukkan bahwa limitnya pada batas-batas yang dihitung.

```
>iterate("g",[~1~;~5~],4)
```

```
  Interval 2 x 5 matrix

  ~0.99999999999999778,1.00000000000000022~      ...
  ~4.99999999999999911,5.00000000000000089~     ...
```

Iterasi berikut konvergen sangat lambat.

$$x_{n+1} = \sqrt{x_n}.$$

```
>iterate("sqrt(x)",2,10)
```

```
  [2,  1.41421,  1.18921,  1.09051,  1.04427,  1.0219,  1.01089,
  1.00543,  1.00271,  1.00135,  1.00068]
```

Kekonvergenan iterasi tersebut dapat dipercepatdengan percepatan Steffenson:

```
>steffenson("sqrt(x)",2,10)
```

```
  [1.04888,  1.00028,  1,  1]
```

Iterasi menggunakan Loop yang ditulis Langsung

Berikut adalah beberapa contoh penggunaan loop untuk melakukan iterasi yang ditulis langsung pada baris perintah.

```
>x=2; repeat x=(x+2/x)/2; until x^2~=2; end; x,
```

```
  1.41421356237
```

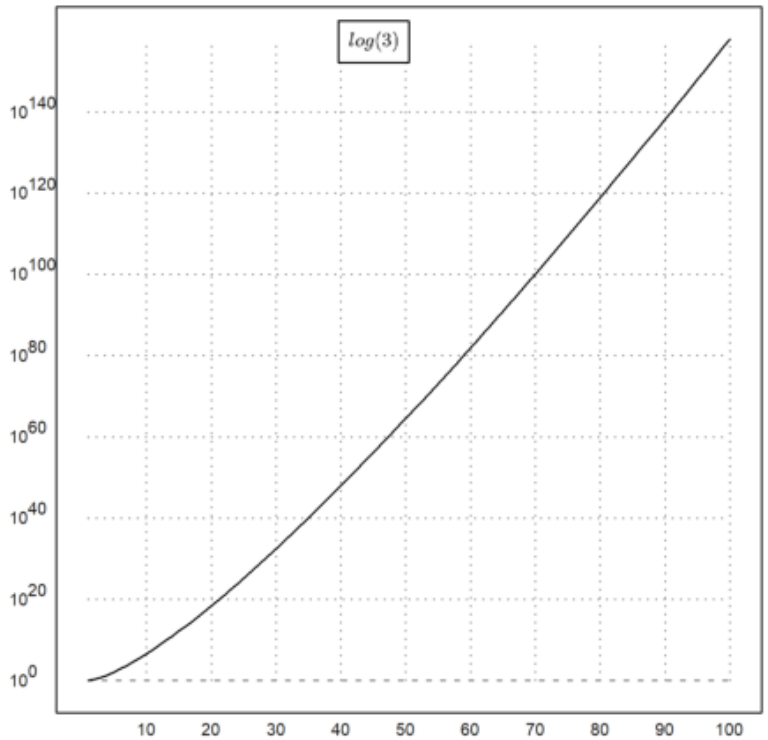
Penggabungan matriks menggunakan tanda "|" dapat digunakan untuk menyimpan semua hasil iterasi.

```
>v=[1]; for i=2 to 8; v=v|(v[i-1]*i); end; v,
```

```
  [1,  2,  6,  24,  120,  720,  5040,  40320]
```

hasil iterasi juga dapat disimpan pada vektor yang sudah ada.

```
>v=ones(1,100); for i=2 to cols(v); v[i]=v[i-1]*i; end; ...
>plot2d(v,logplot=1); textbox(latex(&log(n)),x=0.5):
```




```
>A =[0.5,0.2;0.7,0.1]; b=[2;2]; ...
>x=[1;1]; repeat xnew=A.x-b; until all(xnew~=x); x=xnew; end; ...
>x,
```

```
       -7.09677
       -7.74194
```

Iterasi di dalam Fungsi

Fungsi atau program juga dapat menggunakan iterasi dan dapat digunakan untuk melakukan iterasi. Berikut adalah beberapa contoh iterasi di dalam fungsi.

Contoh berikut adalah suatu fungsi untuk menghitung berapa lama suatu iterasi konvergen. Nilai fungsi tersebut adalah hasil akhir iterasi dan banyak iterasi sampai konvergen.

```
>function map hiter(f$,x0) ...
```

```
  x=x0;
  maxiter=0;
  repeat
    xnew=f$(x);
    maxiter=maxiter+1;
    until xnew~=x;
    x=xnew;
  end;
  return maxiter;
  endfunction
```

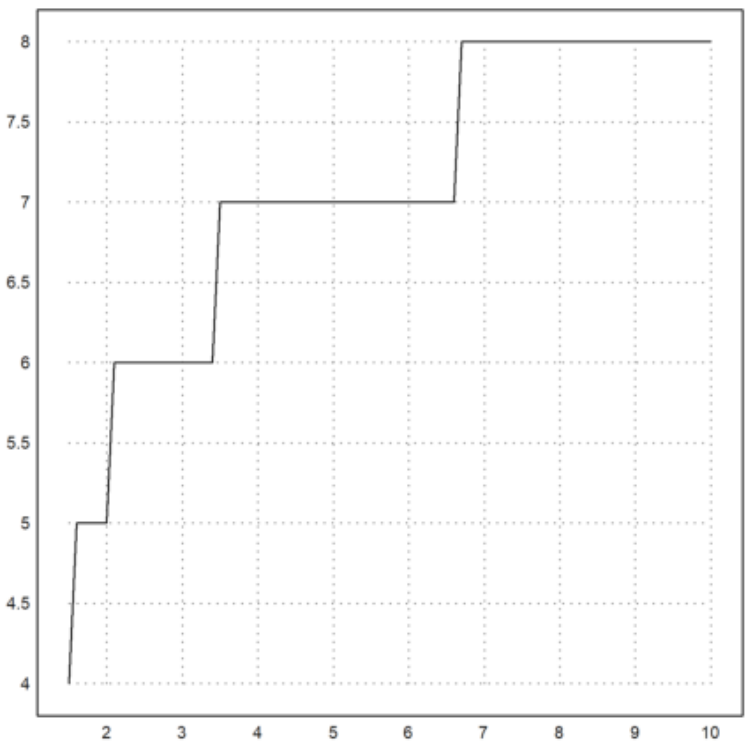
Misalnya, berikut adalah iterasi untuk mendapatkan hampiran akar kuadrat 2, cukup cepat, konvergen pada iterasi ke-5, jika dimulai dari hampiran awal 2.

```
>hiter("(x+2/x)/2",2)
```

```
  5
```

Karena fungsinya didefinisikan menggunakan "map". maka nilai awalnya dapat berupa vektor.

```
>x=1.5:0.1:10; hasil=hiter("(x+2/x)/2",x); ...
>  plot2d(x,hasil):
```



Dari gambar di atas terlihat bahwa kekonvergenan iterasinya semakin lambat, untuk nilai awal semakin besar, namun penambahnnya tidak kontinu. Kita dapat menemukan kapan maksimum iterasinya bertambah.

```
>hasil[1:10]
```

```
  [4,  5,  5,  5,  5,  5,  6,  6,  6,  6]
```

```
>x[nonzeros(differences(hasil))]
```

```
  [1.5,  2,  3.4,  6.6]
```

maksimum iterasi sampai konvergen meningkat pada saat nilai awalnya 1.5, 2, 3.4, dan 6.6.
Contoh berikutnya adalah metode Newton pada polinomial kompleks berderajat 3.

```
>maxima p &= x^3-1; newton &= x-p/diff(p,x); $newton
```

$$\text{newton}$$

Selanjutnya didefinisikan fungsi untuk melakukan iterasi (aslinya 10 kali).

```
>function iterasi(f$,x,n=10) ...
```

```
endfunction
loop 1 to n; x=f$(x); end;
return x;
endfunction
```

Kita mulai dengan menentukan titik-titik grid pada bidang kompleksnya.

```
>maxima r=1.5; x=linspace(-r,r,501); Z=x+I*x'; W=iterasi(newton,Z);
```

Berikut adalah akar-akar polinomial di atas.

```
>z=&solve(p)()
```

```
0
```

Untuk menggambar hasil iterasinya, dihitung jarak dari hasil iterasi ke-10 ke masing-masing akar, kemudian digunakan untuk menghitung warna yang akan digambar, yang menunjukkan limit untuk masing-masing nilai awal.
Fungsi plotrgb() menggunakan jendela gambar terkini untuk menggambar warna RGB sebagai matriks.

```
>C=rgb(max(abs(W-z[1]),1),max(abs(W-z[2]),1),max(abs(W-z[3]),1)); ...
>  plot2d(none,-r,r,-r,r); plotrgb(C):
```

```
Variable W not found!
Error in:
C=rgb(max(abs(W-z[1]),1),max(abs(W-z[2]),1),max(abs(W-z[3]),1) ...
                     ^
```

Iterasi Simbolik

Seperti sudah dibahas sebelumnya, untuk menghasilkan barisan ekspresi simbolik dengan Maxima dapat digunakan fungsi makelist().

```
>maxima &powerdisp:true // untuk menampilkan deret pangkat mulai dari suku berpangkat terk
```

```
true
```

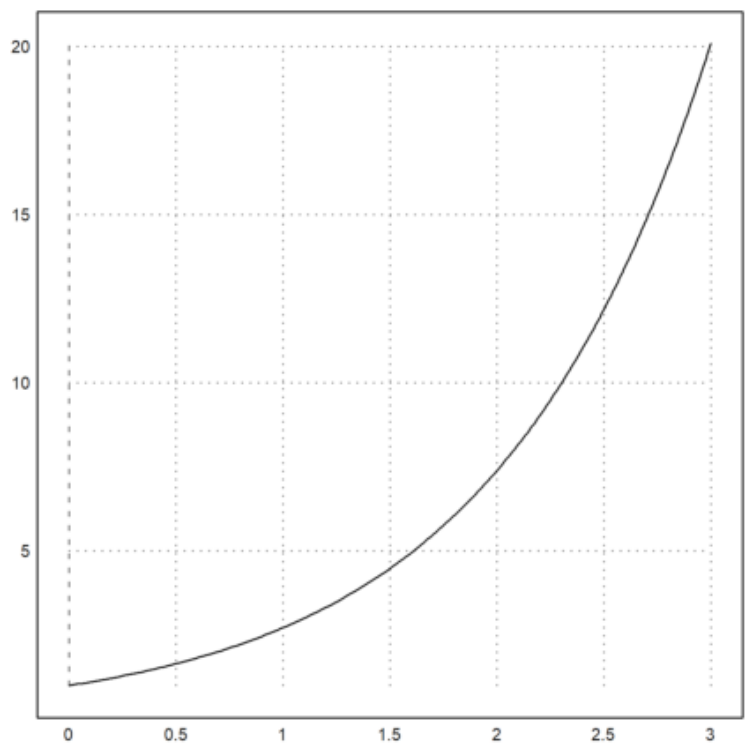
```
>deret &= makelist(taylor(exp(x),x,0,k),k,1,3); $deret // barisan deret Taylor untuk e^x
```

```
Maxima said:
taylor: 0.1539740213994798*r cannot be a variable.
 -- an error. To debug this try: debugmode(true);

Error in:
deret &= makelist(taylor(exp(x),x,0,k),k,1,3); $deret // baris ...
                                               ^
```

Untuk mengubah barisan deret tersebut menjadi vektor string di EMT digunakan fungsi mxm2str(). Selanjutnya, vektor string/ekspresi hasilnya dapat digambar seperti menggambar vektor eskpresi pada EMT.

```
>plot2d("exp(x)",0,3): // plot fungsi aslinya, e^x
```




```
>plot2d(mxm2str("deret"),>add,color=4:6): // plot ketiga deret taylor hampiran fungsi ters
```

```
Maxima said:
length: argument cannot be a symbol; found deret
 -- an error. To debug this try: debugmode(true);

mxmeval:
    return evaluate(mxm(s));
Try "trace errors" to inspect local variables after errors.
mxm2str:
    n=mxmeval("length(VVV)");
```

Selain cara di atas dapat juga dengan cara menggunakan indeks pada vektor/list yang dihasilkan.

```
>$deret[3]
```

$$deret_3$$

```
>plot2d(["exp(x)",&deret[1],&deret[2],&deret[3]],0,3,color=1:4):
```

```
Cannot assign a string to real!
%ploteval:
    y[:,1]=y0;
Try "trace errors" to inspect local variables after errors.
plot2d:
    u=u_(%ploteval(xx[#],t;args()));
```

```
>$sum(sin(k*x)/k,k,1,5)
```

$$\frac{\sin(0.1495626233385798\,r)}{5}, \sin(0.1495626233385798\,r) + \frac{\sin(0.2990052590166597\,r)}{2} + \frac{\sin(0.4485078885240895\,r)}{3} + \frac{\sin(0.5980105180321194\,r)}{4} + \sin$$

Berikut adalah cara menggambar kurva

$$y = \sin(x) + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots.$$

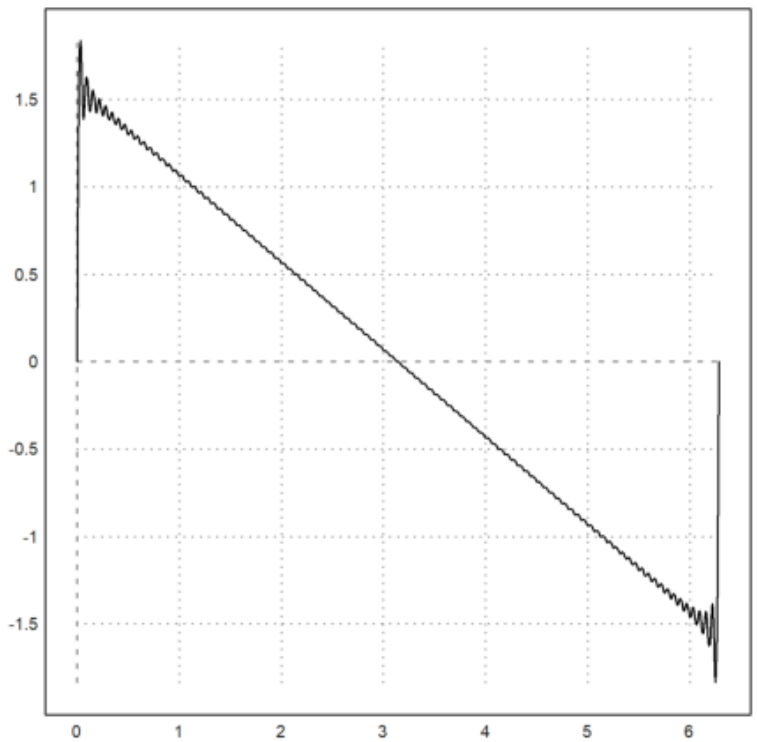
```
>plot2d(&sum(sin((2*k+1)*x)/(2*k+1),k,0,20),0,2pi):
```

```
Maxima output too long!
Error in:
plot2d(&sum(sin((2*k+1)*x)/(2*k+1),k,0,20),0,2pi): ...
                                            ^
```

Hal serupa juga dapat dilakukan dengan menggunakan matriks, misalkan kita akan menggambar kurva

$$y = \sum_{k=1}^{100} \frac{\sin(kx)}{k}, \quad 0 \le x \le 2\pi.$$

```
>x=linspace(0,2pi,1000); k=1:100; y=sum(sin(k*x')/k)'; plot2d(x,y):
```



Tabel Fungsi

Terdapat cara menarik untuk menghasilkan barisan dengan ekspresi Maxima. Perintah mxmtable() berguna untuk menampilkan dan menggambar barisan dan menghasilkan barisan sebagai vektor kolom.

Sebagai contoh berikut adalah barisan turunan ke-n x^x di x=1.

```
>maxima mxmtable("diffat(x^x,x=1,n)","n",1,8,frac=1)
```

```
mxmtable(diffat(x^x,x=1,n), n, 1, 8, frac = 1)
```

```
>$'sum(k, k, 1, n) = factor(ev(sum(k, k, 1, n),simpsum=true)) // simpsum:menghitung deret
```

$$\sum_{k=1}^{3} k = 6$$

```
>$'sum(1/(3^k+k), k, 0, inf) = factor(ev(sum(1/(3^k+k), k, 0, inf),simpsum=true))
```

$$\sum_{k=0}^{\infty} \frac{1}{k+3^k} = \sum_{k=0}^{\infty} \frac{1}{k+3^k}$$

Di sini masih gagal, hasilnya tidak dihitung.

```
>$'sum(1/x^2, x, 1, inf)= ev(sum(1/x^2, x, 1, inf),simpsum=true) // ev: menghitung nilai e
```

$$\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$$

```
>$'sum((-1)^(k-1)/k, k, 1, inf) = factor(ev(sum((-1)^(x-1)/x, x, 1, inf),simpsum=true))
```

$$\sum_{k=1}^{\infty} \frac{(-1)^{-1+k}}{k} = -\sum_{x=1}^{\infty} \frac{(-1)^x}{x}$$

Di sini masih gagal, hasilnya tidak dihitung.

```
>$'sum((-1)^k/(2*k-1), k, 1, inf) = factor(ev(sum((-1)^k/(2*k-1), k, 1, inf),simpsum=true)
```

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{-1+2\,k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{-1+2\,k}$$

```
>$ev(sum(1/n!, n, 0, inf),simpsum=true)
```

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

Di sini masih gagal, hasilnya tidak dihitung, harusnya hasilnya e.

```
>maxima &assume(abs(x)<1); $'sum(a*x^k, k, 0, inf)=ev(sum(a*x^k, k, 0, inf),simpsum=true),
```

```
rat: replaced 1.66665833335744e-7 by 15819/94914474571 = 1.66665833335744e-7
Is - 94914474571 + 15819 r positive, negative or zero?
```

Deret geometri tak hingga, dengan asumsi rasional antara -1 dan 1.

```
>$'sum(x^k/k!,k,0,inf)=ev(sum(x^k/k!,k,0,inf),simpsum=true)
```

$$\left[0, \sum_{k=0}^{\infty} \frac{\left(1.66665833335744 \times 10^{-7}\right)^k r^k}{k!}, \sum_{k=0}^{\infty} \frac{\left(1.33330666692022 \times 10^{-6}\right)^k r^k}{k!}, \sum_{k=0}^{\infty} \frac{\left(4.499797504338432 \times 10^{-6}\right)^k r^k}{k!}, \sum_{k=0}^{\infty} \frac{\left(1.0\right.}{}\right.$$

```
>maxima $limit(sum(x^k/k!,k,0,n),n,inf)
```

```
limit: second argument must be a variable, not a constant; found: 3
 -- an error. To debug this try: debugmode(true);
```

```
>function d(n) &= sum(1/(k^2-k),k,2,n); $'d(n)=d(n)
```

$$d(3) = \frac{2}{3}$$

```
>$d(10)=ev(d(10),simpsum=true)
```

$$\frac{2}{3} = \frac{2}{3}$$

```
>$d(100)=ev(d(100),simpsum=true)
```

$$\frac{2}{3} = \frac{2}{3}$$

Deret Taylor

Deret Taylor suatu fungsi f yang diferensiabel sampai tak hingga di sekitar x=a adalah:

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k f^{(k)}(a)}{k!}.$$

```
>maxima $'e^x =taylor(exp(x),x,0,10) // deret Taylor e^x di sekitar x=0, sampai suku ke-11
```

```
taylor: 0.1539740213994798 r cannot be a variable.
 -- an error. To debug this try: debugmode(true);
```

```
>maxima $'log(x)=taylor(log(x),x,1,10)// deret log(x) di sekitar x=1
```

```
log: encountered log(0).
 -- an error. To debug this try: debugmode(true);
```

Sekian. Terima kasih.