A VISUAL APPROACH TO BASIC CONCEPTS OF CALCULUS

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A typical problem in learning and teaching calculus lies in the fact that often students calculate without comprehending the basic ideas of differential and integral calculus. This issue can be addressed by the use of technology to create a suitable learning environment for students to first experience and understand the basic ideas of differential and integral calculus, before then proceeding to calculate fractions, sums and limits! The overhead-slide sequence 'Funktionen-mikroskop' (lit 'Function Microscope') and the mechanical device 'Integraph' are classic approaches which we can revive today in creating dynamic learning environments with the help of modern software, such as the digital Function Loupe and the digital Integrator as briefly discussed here.

THE 'FUNKTIONENMIKROSKOP'

The *Funktionenmikroskop* is a learning aid developed by A. Kirsch (1979, 1980) for the visualisation of the basic idea of derivation. He created a sequence of overhead-slides with a given function f and a given point A on the graph of f. He successively magnified the surroundings of A until the visible part of the graph looked like a straight line. At this point, the slope of the straight line will be the same as the slope of the function-graph. This method was an exclusively graphical and visual approach without the use of any calculations. The restrictions of the *Funktionenmikroskop* were that the given function f and the given point A could not be changed. Later versions with digital function plotters avoided these problems, but they were also limited, because by zooming in they would show local information but at the same time would lose the overview, and hence the global information.

THE DIGITAL FUNCTION LOUPE

The digital Function Loupe (Elschenbroich, Seebach & Schmidt, 2014; Elschenbroich 2015a) is a learning environment, which uses the two graphic windows of GeoGebra. It provides us with the combination of both, the global view and the local view: One window displays the standard-zoom and the second window is used for interactive zooming. The size of a square around point A in the global window can be varied by a slider h (i.e. the loupe controller) and the content of this square will be magnified in the second window. In the global view, additionally to point A = (x, f(x)), we use points A_l = (x-h; f(x-h)) and A_r = (x+h; f(x+h)) for our investigation. The straight line through A_l and A is the left-hand secant and the straight line through A and A_r is the right-hand secant and we can then measure each slope.

We can vary the position of the loupe by dragging point A on the graph of the function f and we can vary the size of the square by minimizing h at the slider. If h is small enough, the graph in the local window will look like a straight line (if function f is 'well behaved', i.e. differentiable).

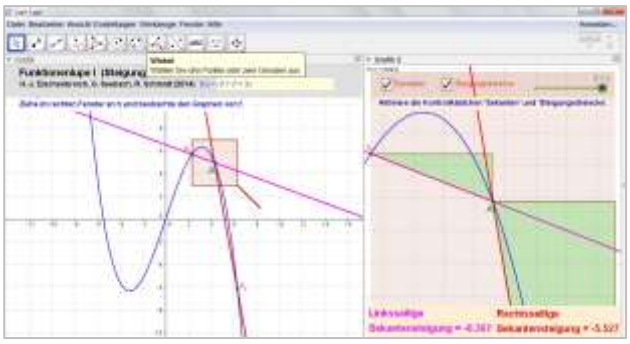
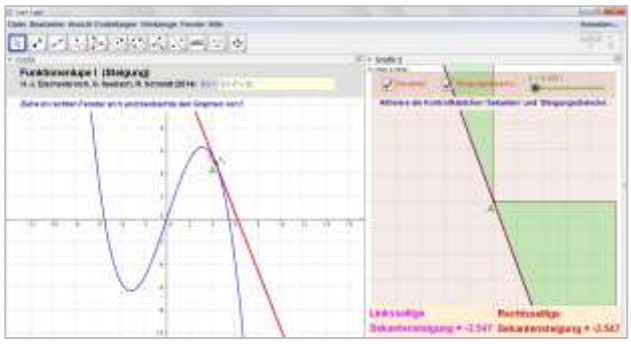

Figure 1a, b: Function Loupe I, h = 2 and h = 0.0001

The software allows us furthermore to transfer the local left-hand and right-hand slopes in the y-coordinates of point P_l and point P_r. When we drag at A these points create a locus, which will be the graphs of the left-hand secant slope-function and the right-hand secant slope-function - without knowing the terms of these functions! For small h these graphs will coincide (i.e. on the global window no difference is visible) and what we will see in the local window is the graph of the tangent slope-function, the derivative!

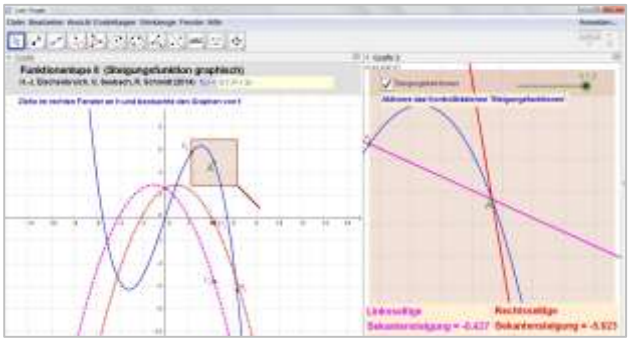
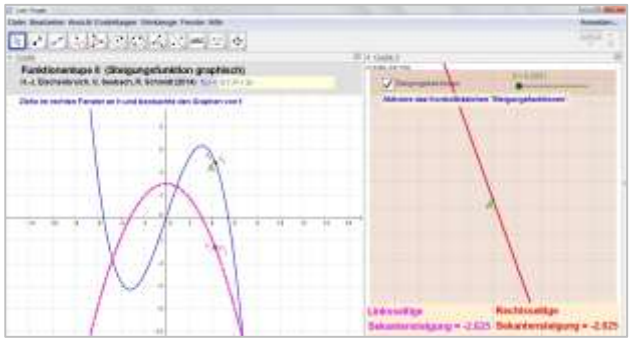

Figure 2a, b: Function-Loupe II, h = 2 and h = 0.0001

A further step is not just defining two straight lines through A_l, A and A_r, but drawing a circle through these three points. This allows us a visual and graphical approach to the curvature of the graph of function f at the point A, because the curvature of a slope is 1/radius. For small h we cannot observe any variation of the midpoint and the radius of the circle anymore. This delivers a visual approach to the osculating circle and thereby to the curvature of the function graph!

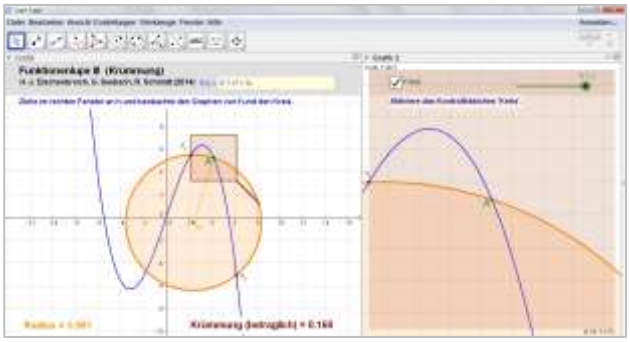
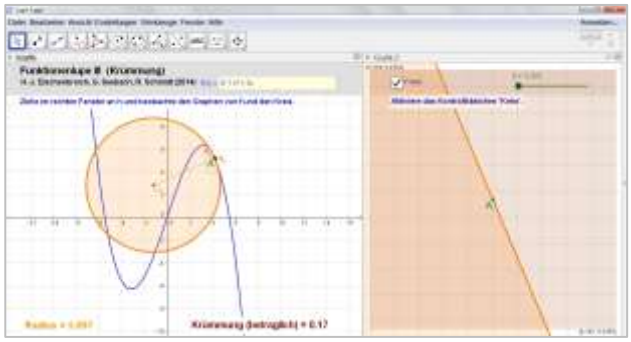

Figure 3a, b: Function-Loupe III, h = 2 and h = 0.0001

Further implementations with TI-Nspire and Cinderella are available (www.funktionenlupe.de).

THE INTEGRIMETER AND THE INTEGRAPH

For a further application of modern digital learning I have looked at the *Integrimeter* and *Integraph*. These are special mechanical devices formerly employed in technical measurement (Rechnerlexikon, 2015). The *Integrimeter* will measure integrals of graphically defined functions and the *Integraph* is able to plot integral curves of such functions. With the *Integraph* we drive an input stylus S on the graph of a function f and an ingenious linkage of bars will create a curve F with an output stylus Z in such a way that the tangent at the graph of F always has the slope f(x).

THE DIGITAL INTEGRATOR

The Integrator is a digital learning environment inspired by the devices *Integrimeter* and *Integraph*. Firstly, we can use the Integrator in a classical way for calculating lower and upper sums. With a slider n we can vary the number of subdivision of the interval [a; b]. For large n, we see that the upper sum and the lower sum will - slowly - converge.


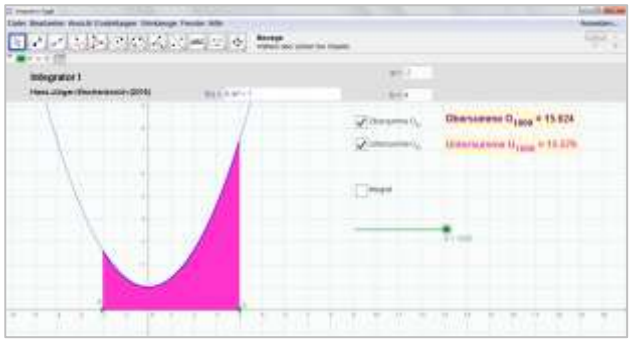

Figure 4a, b: Integrator I, n = 8 and n = 1000

Secondly, we can now vary a point S on the graph of f over the interval [a; b] and define two points U_a and O_a with the x-coordinate x(S) and the y-coordinate lower sum and upper sum, respectively. When we drag at S, each of these points will create a locus. These loci will represent the graphs of the lower sum-function and the upper sum-function without knowing the terms of these functions! For large n, these graphs will coincide (i.e. on the screen there is no longer any difference visible) and so we get the graph of the integral function F (Elschenbroich, 2015b).

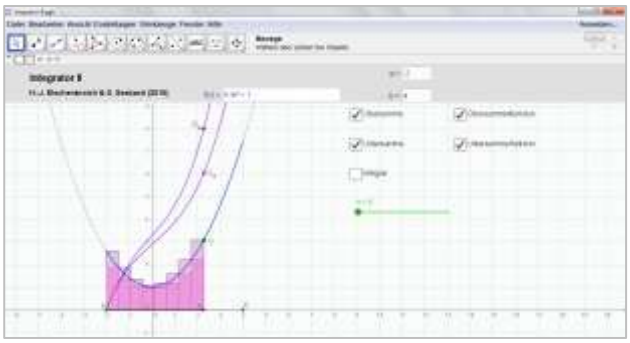
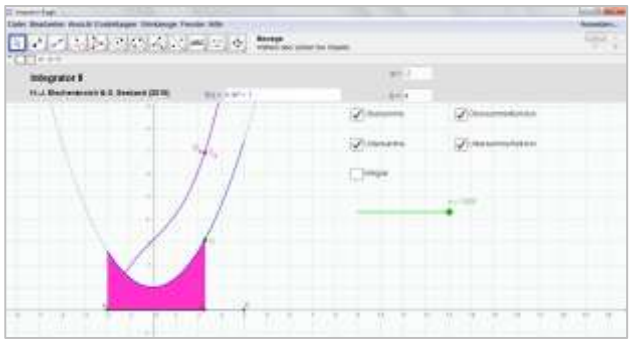

Figure 5a, b: Integrator II, n = 8 and n = 1000

Lastly, we can visualize that the slope of the integral function F at x is equal to the value of f(x). This means, that the integral function F - created as a locus by point Z = (x; $\int_a^x f(t)dt$) - is an antiderivative of function f, which is the substance of the fundamental theorem of calculus!

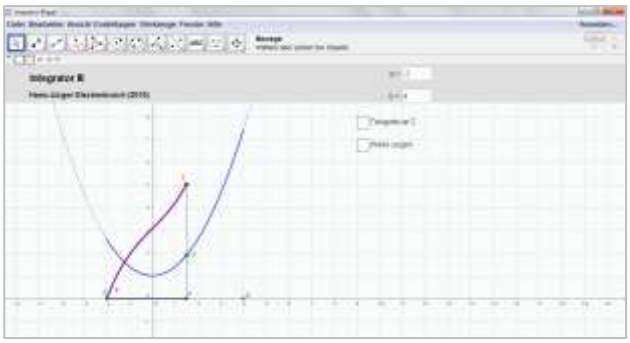
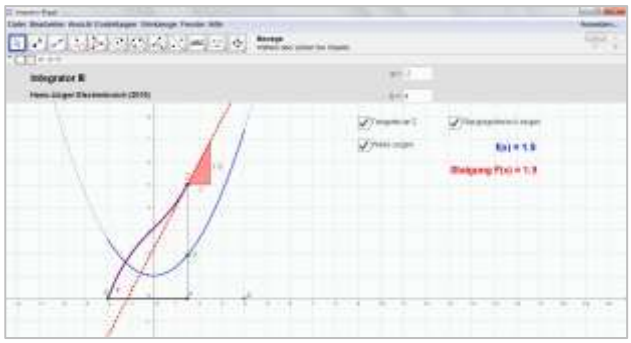

Figure 6a, b: Integrator III, visualisation of the fundamental theorem of calculus

CONCLUSION

The substantial problem in teaching the first steps of integral and differential calculus lies predominantly in the fact that students are 'drilled' to execute calculus as a routine without first comprehending the underlying concepts, i.e., what these calculations actually mean and why they are undertaking them. The ideas presented in this paper are designed to overcome this issue by creating a visualized, graphical and a calculus-free introduction to basic concepts like slope, derivation, curvature, integral and anti-derivative with 'well behaved' functions.

Function-loupe and Integrator are digital learning environments that aid the student - as first learning steps - to comprehend the basics of integral and differential theory without requiring any calculation at the user level. They are not intended to substitute the calculations and the teaching of the theory of infinitesimal calculus, but deliver a solid foundation upon which to build an understanding of differential and integral calculus.

References

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