In[1]:=
```
(* 6-Eck-Netze aus den Kreisen von 3 Kreisbüscheln ,
     euklidische Basis  11.02.2022  W.F. *)
p0 := 1 / 2 * {-I, 0, 1}
p∞ := {I, 0, 1}
g0 := {0, -I, 0}
p[u_] := u^2 / 2 p∞ + u * g0 + p0
p::usage = "Ordnet der komplexen Zahl u die Berührgerade p[u] zu";
c[u_, v_] := Cross[p[u], p[v]]
c::usage = "Ordnet den komplexe Zahlen u,v das Kreuzprodukt [p[u],p[v]] zu";
cg[u_, v_] := 1 / (u - v) * (u * v * p∞ + (u + v) * g0 + 2 * p0)
cg::usage = "die normierte Verbindungsgerade";
{p0, g0, p∞, p[z], ComplexExpand[p[x + I * y].p[x + I * y]]}
q[x_, y_] := ComplexExpand[p[x + I * y]]
q::usage = "Berührgerade zu z=x+iy, x,y als reelle Variablen";
Factor[q[x, y]]
Det[{p∞, g0, p0}]
```

Out[10]= $\left\{\left\{-\frac{i}{2}, 0, \frac{1}{2}\right\}, \{0, -i, 0\}, \{i, 0, 1\}, \left\{-\frac{i}{2} + \frac{i z^2}{2}, -i z, \frac{1}{2} + \frac{z^2}{2}\right\}, 0\right\}$

Out[13]= $\left\{\frac{1}{2} i (-1 + x + i y) (1 + x + i y), -i x + y, \frac{1}{2} (-i + x + i y) (i + x + i y)\right\}$

Out[14]= 1

In[15]:=
```
(*euklidische Basis *)
m := {p∞, g0, p0}
MatrixForm[m]
```

Out[16]//MatrixForm=

$$\begin{pmatrix} i & 0 & 1 \\ 0 & -i & 0 \\ -\frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

In[17]:=
```
(* Skalarprodukt-Tabelle *)
MatrixForm[Table[m[[i]].m[[j]], {i, 3}, {j, 3}]]
```

Out[17]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In[18]:=
```
(* LIE-Produkt-Tabelle *)
{MatrixForm[Table[Cross[m[[i]], m[[j]]], {i, 3}, {j, 3}]],
 MatrixForm[{{0, "p0", "g0"}, {"-p0", 0, "p∞"}, {"-g0", "-p∞", 0}}]}
```

Out[18]=

$$\left\{\begin{pmatrix} \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}i\\0\\1\end{pmatrix} & \begin{pmatrix}0\\-i\\0\end{pmatrix} \\ \begin{pmatrix}-i\\0\\-1\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}-\frac{i}{2}\\0\\\frac{1}{2}\end{pmatrix} \\ \begin{pmatrix}0\\i\\0\end{pmatrix} & \begin{pmatrix}\frac{i}{2}\\0\\-\frac{1}{2}\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} \end{pmatrix}, \begin{pmatrix} 0 & p0 & g0 \\ -p0 & 0 & p\infty \\ -g0 & -p\infty & 0 \end{pmatrix}\right\}$$

In[19]:=
```
(*Verbindungsgeraden [1,-1], [1,i],[i,-1]*)
vg1 := cg[1, -1]
vg2 := cg[1, I]
vg3 := cg[I, -1]
vgi := cg[I, -I]
{vg1, vg2, vg3, vgi}
```

Out[23]= {{-i, 0, 0}, {-i, 1, i}, {-i, 1, -i}, {0, 0, -i}}

In[24]:=
```
qu2[g1_, g2_, x_, y_] := ComplexExpand[Im[g1.q[x, y] * Conjugate[g2.q[x, y]]]]
qu2::usage = "Hemitesche Form g1∧g2(p(z),p(z))";
qu3[g1_, g2_, g3_, x_, y_] :=
 ComplexExpand[qu2[Cross[g2, g1], g3, x, y] + qu2[g2, Cross[g3, g1], x, y]]
qu3::usage = "Hermitesche Form [g1,g2∧g3]= [g2,g1]∧g3+g2∧[g3,g1]";
```

In[28]:=
```
Factor[qu2[g0, vg1, x, y]]
```

Out[28]= $\frac{1}{2}\,y\,\left(1+x^2+y^2\right)$

In[29]:=
```
bed1[g1_, g2_, g3_, x_, y_] := ComplexExpand[
   qu3[g1, g1, g2, x, y] / qu2[g1, g2, x, y] * qu3[g2, g2, g3, x, y] / qu2[g2, g3, x, y] -
    qu3[g2, g1, g2, x, y] / qu2[g1, g2, x, y] * qu3[g1, g3, g1, x, y] / qu2[g3, g1, x, y]]
```

In[30]:=
```
bed3[g1_, g2_, g3_, x_, y_] := ComplexExpand[
   qu3[g2, g2, g3, x, y] / qu2[g2, g3, x, y] * qu3[g1, g2, g3, x, y] / qu2[g2, g3, x, y] -
    qu3[g1, g3, g1, x, y] / qu2[g3, g1, x, y] * qu3[g2, g3, g1, x, y] / qu2[g3, g1, x, y]]
bed2[g1_, g2_, g3_, x_, y_] := ComplexExpand[
   (qu2[Cross[Cross[g3, g1], g2], g1, x, y] + qu2[Cross[g3, g1], Cross[g1, g2], x, y]) /
     qu2[g3, g1, x, y] - (qu2[Cross[g2, g1], Cross[g3, g2], x, y] +
        qu2[g2, Cross[Cross[g3, g2], g1], x, y]) / qu2[g2, g3, x, y]]
```

In[32]:=
```
bed6Eck[g1_, g2_, g3_, x_, y_] :=
 bed1[g1, g2, g3, x, y] + bed2[g1, g2, g3, x, y] + bed3[g1, g2, g3, x, y]
bed6Eck::usage = "Sechseckbedingung für 3 Infinitesimale g1,g2,g3";
listBed6Eck[g1_, g2_, g3_, x_, y_] :=
 {bed1[g1, g2, g3, x, y], bed2[g1, g2, g3, x, y], bed3[g1, g2, g3, x, y]}
```

In[35]:=
```
(* Fall 2: kommutative Infinitesimale *)
qu2[Exp[I * α] * vg1, I * vg1, x, y]
Simplify[bed6Eck[Exp[I * α] * vg1, Exp[I * β] * vg1, Exp[I * γ] vg1, x, y]]
Simplify[listBed6Eck[Exp[I * α] * vg1, Exp[I * β] * vg1, Exp[I * γ] * vg1, x, y]]
```

Out[35]= $-\frac{\text{Cos}[\alpha]}{4} + \frac{1}{2} x^2 \text{Cos}[\alpha] - \frac{1}{4} x^4 \text{Cos}[\alpha] - \frac{1}{2} y^2 \text{Cos}[\alpha] - \frac{1}{2} x^2 y^2 \text{Cos}[\alpha] - \frac{1}{4} y^4 \text{Cos}[\alpha]$

Out[36]= 0

Out[37]= {0, 0, 0}

In[38]:=
```
(* Fall 3: 2 Pole Verbindungsgerade *)
Factor[ComplexExpand[bed6Eck[p0, g0, p∞, x, y]]]
Simplify[listBed6Eck[p0, g0, p∞, x, y]]
Factor[ComplexExpand[bed6Eck[p0, I * g0, p∞, x, y]]]
Simplify[listBed6Eck[p0, I * g0, p∞, x, y]]
Factor[ComplexExpand[bed6Eck[Exp[I * α] * p0, I * g0, p∞, x, y]]]
(* für i*g0: die parabolischen müssen einen Kreis gemeinsam haben *)
(* aber siehe Fall 4! *)
Factor[ComplexExpand[bed6Eck[I * p0, I * g0, p∞, x, y]]]
(* kein 6-Eck-Netz *)
```

Out[38]= 0

Out[39]= $\left\{-\frac{x^2 + y^2}{2 x}, \frac{3 x^2 + y^2}{2 x}, -x\right\}$

Out[40]= 0

Out[41]= $\left\{0, \frac{x^2 + 3 y^2}{2 y}, -\frac{x^2 + 3 y^2}{2 y}\right\}$

Out[42]= $-\Big(\Big(\big(x^2 + y^2\big) \text{Sin}[\alpha] \big(2 x^6 \text{Cos}[\alpha]^3 + 6 x^2 y^4 \text{Cos}[\alpha]^3 - 6 x^5 y \text{Cos}[\alpha]^2 \text{Sin}[\alpha] + 12 x^3 y^3 \text{Cos}[\alpha]^2 \text{Sin}[\alpha] - 6 x y^5 \text{Cos}[\alpha]^2 \text{Sin}[\alpha] + x^6 \text{Cos}[\alpha] \text{Sin}[\alpha]^2 + 15 x^4 y^2 \text{Cos}[\alpha] \text{Sin}[\alpha]^2 - 9 x^2 y^4 \text{Cos}[\alpha] \text{Sin}[\alpha]^2 + y^6 \text{Cos}[\alpha] \text{Sin}[\alpha]^2 - 8 x^3 y^3 \text{Sin}[\alpha]^3\big)\Big) \Big/ \Big(2 x^2 (-x \text{Cos}[\alpha] + y \text{Sin}[\alpha]) \big(-2 x y \text{Cos}[\alpha] - x^2 \text{Sin}[\alpha] + y^2 \text{Sin}[\alpha]\big)^2\Big)\Big)$

Out[43]= $\frac{4 x y^2 (x^2 + y^2)}{(x - y)^2 (x + y)^2}$

In[44]:=
```
(* Fall 4: 2 Pole, gilt nur für g0 elliptisch!*)
Factor[ComplexExpand[bed6Eck[Exp[I * α] * p0, g0, p∞, x, y]]]
Simplify[listBed6Eck[Exp[I * α] * p0, g0, p∞, x, y]]
```

Out[44]= 0

Out[45]=
$$\left\{\frac{y\left(x^2+y^2\right)}{-2\,x\,y\,\mathrm{Cos}[\alpha]+\left(-x^2+y^2\right)\mathrm{Sin}[\alpha]},\ \frac{-2\,y^2\left(3\,x^2+y^2\right)\mathrm{Cos}[\alpha]^2+8\,x\,y^3\,\mathrm{Cos}[\alpha]\,\mathrm{Sin}[\alpha]+\left(x^4-5\,y^4\right)\mathrm{Sin}[\alpha]^2}{2\,y\left(-2\,x\,y\,\mathrm{Cos}[\alpha]+\left(-x^2+y^2\right)\mathrm{Sin}[\alpha]\right)},\ \frac{1}{2}\left(-2\,x\,\mathrm{Cos}[\alpha]+\frac{x^2\,\mathrm{Sin}[\alpha]}{y}+3\,y\,\mathrm{Sin}[\alpha]\right)\right\}$$

In[46]:=
```
(* Fall 5 auch 2 Pole , mit Loxodromen *)
Simplify[bed6Eck[Exp[I * α] * g0, g0, p0, x, y]]
Simplify[listBed6Eck[Exp[I * α] * g0, g0, p0, x, y]]
```

Out[46]= 0

Out[47]=
$$\left\{0,\ \frac{\left(x^2+y^2\right)\mathrm{Sin}[\alpha]^2}{y\,(y\,\mathrm{Cos}[\alpha]-x\,\mathrm{Sin}[\alpha])},\ -\frac{\left(x^2+y^2\right)\mathrm{Sin}[\alpha]^2}{y\,(y\,\mathrm{Cos}[\alpha]-x\,\mathrm{Sin}[\alpha])}\right\}$$

In[48]:=
```
(* Fall 6  auch 2 Pole *)
Simplify[bed6Eck[I * p0, I * g0, p0, x, y]]
Simplify[listBed6Eck[I * p0, I * g0, p0, x, y]]
```

Out[48]= 0

Out[49]=
$$\left\{\frac{x^2+y^2}{2\,x},\ -\frac{x^2+y^2}{2\,x},\ 0\right\}$$

In[50]:=
```
(* Fall 7 & 8:  Drei Infinitesimale mit je einem gemeinsamen Pol,
    einer oder alle 3 (gleichwinklige) Loxodrome *)
Simplify[bed6Eck[Exp[I * α] * vg1, vg2, vg3, x, y]]
Simplify[listBed6Eck[Exp[I * α] * vg1, vg2, vg3, x, y]];
(* sehr lange Terme und längere Rechenzeiten*)
Simplify[bed6Eck[Exp[I * α] * vg1, Exp[I * α] * vg2, Exp[I * α] * vg3, x, y]]
Simplify[listBed6Eck[Exp[I * α] * vg1, Exp[I * α] * vg2, Exp[I * α] * vg3, x, y]]
```

Out[50]= 0

Out[52]= 0

Out[53]=

$$\Big\{\frac{(1-2x+x^2+y^2)\,\mathrm{Sin}[2\alpha]}{-1+x^2+y^2},\ \Big(x^6-4x^3y-4xy(-1+y^2)+3x^4(-1-2y+y^2)+(-1+y)^3(1-3y-3y^2+y^3)+3x^2(1+2y^2-4y^3+y^4)-3\big(x^6+x^4(-1-2y+3y^2)+(-1+y-y^2+y^3)^2+x^2(-1+4y+2y^2-4y^3+3y^4)\big)\,\mathrm{Sin}[2\alpha]\Big)\Big/\Big((x^2+(-1+y)^2)(-1+x^2+y^2)(1+2x+x^2+y^2)\Big),$$

$$\Big(-x^6+4x^3y+x^4(3+6y-3y^2)+4xy(-1+y^2)-(-1+y)^3(1-3y-3y^2+y^3)-3x^2(1+2y^2-4y^3+y^4)+2\big(x^6+x^4(-1-2y+3y^2)+(-1+y-y^2+y^3)^2+x^2(-1+4y+2y^2-4y^3+3y^4)\big)\,\mathrm{Sin}[2\alpha]\Big)\Big/\Big((x^2+(-1+y)^2)(-1+x^2+y^2)(1+2x+x^2+y^2)\Big)\Big\}$$

In[54]:=
```
(* Fall 9: 3 Pole -1,1,I : i vg1, vg2, vg3, i*p[i] *)
Simplify[bed6Eck[I * vg1, vg2, vg3, x, y]]
Simplify[listBed6Eck[I * vg1, vg2, vg3, x, y]]
Simplify[bed6Eck[I * vg1, vg2, I * p[I], x, y]]
Simplify[listBed6Eck[I * vg1, vg2, I * p[I], x, y]]
Simplify[bed6Eck[vg2, vg3, I * p[I], x, y]]
Simplify[listBed6Eck[vg2, vg3, I * p[I], x, y]]
```

Out[54]= 0

Out[55]=
$$\Big\{0,\ \Big(2\,\big(x^7 + x^6\,(2-3\,y) + x^5\,(-1+3\,y^2) - (-1+y)^4\,y\,(3+4\,y+3\,y^2) - x^2\,(-1+y)^3\,(2+9\,y+9\,y^2) + x^4\,(-4+3\,y+12\,y^2-9\,y^3) + x^3\,(-1+2\,y^2+3\,y^4) + x\,(1+3\,y^2-8\,y^3+3\,y^4+y^6)\big)\Big) \Big/ \Big(\big(x^4+2\,x^2\,(-1+y)\,y+(-1+y)^3\,(1+y)\big)\,\big(x^4-4\,x\,y+(-1+y)^2\,(1+y^2)+2\,x^2\,(-1-y+y^2)\big)\Big),$$
$$-\Big(\Big(2\,\big(x^7 + x^6\,(2-3\,y) + x^5\,(-1+3\,y^2) - (-1+y)^4\,y\,(3+4\,y+3\,y^2) - x^2\,(-1+y)^3\,(2+9\,y+9\,y^2) + x^4\,(-4+3\,y+12\,y^2-9\,y^3) + x^3\,(-1+2\,y^2+3\,y^4) + x\,(1+3\,y^2-8\,y^3+3\,y^4+y^6)\big)\Big) \Big/ \Big(\big(x^4+2\,x^2\,(-1+y)\,y+(-1+y)^3\,(1+y)\big)\,\big(x^4-4\,x\,y+(-1+y)^2\,(1+y^2)+2\,x^2\,(-1-y+y^2)\big)\Big)\Big)\Big\}$$

Out[56]= 0

Out[57]=
$$\Big\{\frac{1-2\,x+x^2+y^2}{-1+x^2+y^2},\ \Big(-3\,x^7+(-1+y)^6\,y-3\,x\,(-1+y)^4\,(1+y)^2+x^6\,(6+y)+x^5\,(-5+6\,y-9\,y^2) - x^3\,(-1+y)^2\,(5+6\,y+9\,y^2)+3\,x^2\,(-1+y)^2\,(2+y+y^3)+x^4\,(4-9\,y+6\,y^2+3\,y^3)\Big) \Big/ \Big(x\,\big(-2\,x^3+x^4-2\,x\,(-1+y)^2+2\,x^2\,(-1+y)^2+(-1+y)^4\big)\,\big(-1+x^2+y^2\big)\Big),$$
$$\Big(2\,x^7-(-1+y)^6\,y-x^6\,(2+y)+2\,x\,(-1+y)^4\,(1+3\,y+y^2)+2\,x^3\,(-1+y)^2\,(-1+4\,y+3\,y^2) + x^5\,(-2-2\,y+6\,y^2)+x^4\,(4-3\,y+2\,y^2-3\,y^3)-x^2\,(-1+y)^2\,(2+7\,y-4\,y^2+3\,y^3)\Big) \Big/ \Big(x\,\big(-2\,x^3+x^4-2\,x\,(-1+y)^2+2\,x^2\,(-1+y)^2+(-1+y)^4\big)\,\big(-1+x^2+y^2\big)\Big)\Big\}$$

Out[58]= 0

Out[59]=
$$\Big\{0,\ -\frac{8\,x\,\big(x^2+(-1+y)^2\big)}{x^4+(-1+y)^4+2\,x^2\,(-1-2\,y+y^2)},\ \frac{8\,x\,\big(x^2+(-1+y)^2\big)}{x^4+(-1+y)^4+2\,x^2\,(-1-2\,y+y^2)}\Big\}$$

In[60]:=
```
(* Fall 10 ON-Basis Fall: 3 elliptische *)
Simplify[bed6Eck[g0, vg1, vgi, x, y]]
Simplify[listBed6Eck[g0, vg1, vgi, x, y]]
```

Out[60]= 0

Out[61]=
$$\Big\{0,\ \frac{x^4+2\,x^2\,(-3+y^2)+(-1+y^2)^2}{2\,x\,(-1+x^2+y^2)},\ -\frac{x^4+2\,x^2\,(-3+y^2)+(-1+y^2)^2}{2\,x\,(-1+x^2+y^2)}\Big\}$$

In[62]:=
```
(* Fall 10 ON-Basis Fall: 1 hyperbolisch 2 elliptische *)
Simplify[bed6Eck[I * g0, vg1, vgi, x, y]]
Simplify[listBed6Eck[I * g0, vg1, vgi, x, y]]
```

Out[62]= 0

Out[63]=
$$\Big\{-\frac{1+x^4+6\,y^2+y^4+2\,x^2\,(1+y^2)}{2\,y\,(1+x^2+y^2)},\ \frac{1+x^4+6\,y^2+y^4+2\,x^2\,(1+y^2)}{2\,y\,(1+x^2+y^2)},\ 0\Big\}$$

In[64]:=
```
(* Fall 10 ON-Basis Fall: 2 hyperbolische 1 elliptisch *)
Simplify[bed6Eck[gI * g0, I * vg1, vgi, x, y]]
Simplify[listBed6Eck[I * g0, I * vg1, vgi, x, y]]
```

Out[64]= 0

Out[65]= $\left\{0, -\frac{x^4 + 2 x^2 \left(-3 + y^2\right) + \left(-1 + y^2\right)^2}{2 x \left(-1 + x^2 + y^2\right)}, \frac{x^4 + 2 x^2 \left(-3 + y^2\right) + \left(-1 + y^2\right)^2}{2 x \left(-1 + x^2 + y^2\right)}\right\}$

In[66]:=
```
(* Fall 10 ON-Basis Fall: 3 hyperbolische *)
Simplify[bed6Eck[I * g0, I * vg1, I * vgi, x, y]]
Simplify[listBed6Eck[I * g0, I * vg1, I * vgi, x, y]]
```

Out[66]= 0

Out[67]= $\left\{-\frac{x^4 + 2 x^2 \left(-3 + y^2\right) + \left(-1 + y^2\right)^2}{2 x \left(-1 + x^2 + y^2\right)}, \frac{x^4 + 2 x^2 \left(-3 + y^2\right) + \left(-1 + y^2\right)^2}{2 x \left(-1 + x^2 + y^2\right)}, 0\right\}$

```
(* Test: mit Loxodromen zu einer ON-Basis, ein Beispiel *)
Simplify[bed6Eck[(1 + I) * g0, (I - 1) * vg1, (2 + I) * vgi, x, y]]
Simplify[listBed6Eck[(1 + I) * g0, (I - 1) * vg1, (2 + I) * vgi, x, y]]
```

Out[105]=

$$\Big(11 x^{18} - 6 x^{17} y - 12 x^{15} y \left(-9 + 4 y^2\right) + x^{16} \left(-92 + 87 y^2\right) - \left(-1 + y^2\right)^6 \left(y + y^3\right)^2 - 12 x^{13} y \left(23 - 53 y^2 + 14 y^4\right) + 4 x^{14} \left(-217 - 180 y^2 + 75 y^4\right) - 6 x y \left(-1 + y^2\right)^5 \left(1 + 7 y^2 + 7 y^4 + y^6\right) - 12 x^{11} y \left(75 + 50 y^2 - 129 y^4 + 28 y^6\right) - 12 x^9 y^3 \left(425 - 39 y^2 - 165 y^4 + 35 y^6\right) + 4 x^{12} \left(23 - 343 y^2 - 596 y^4 + 147 y^6\right) + x^2 \left(-1 + y^2\right)^4 \left(11 - 124 y^2 - 94 y^4 - 132 y^6 + 3 y^8\right) - 12 x^3 y \left(-1 + y^2\right)^3 \left(-9 - 69 y^2 - 63 y^4 + 9 y^6 + 4 y^8\right) + 2 x^{10} \left(857 + 684 y^2 + 1830 y^4 - 2168 y^6 + 357 y^8\right) + 4 x^4 \left(-1 + y^2\right)^2 \left(-23 + 479 y^2 + 543 y^4 + 512 y^6 - 230 y^8 + 15 y^{10}\right) - 12 x^7 y \left(-75 + 742 y^4 - 196 y^6 - 115 y^8 + 28 y^{10}\right) + 2 x^8 \left(46 - 4695 y^2 + 354 y^4 + 5498 y^6 - 2340 y^8 + 273 y^{10}\right) - 12 x^5 y \left(-23 - 275 y^2 + 522 y^6 - 199 y^8 - 39 y^{10} + 14 y^{12}\right) + 4 x^6 \left(-217 + 296 y^2 - 2987 y^4 - 716 y^6 + 2581 y^8 - 748 y^{10} + 63 y^{12}\right)\Big) \Big/ \left(x \left(-1 + x^2 + y^2\right) \left(-1 + x^4 + 12 x y + 2 x^2 y^2 + y^4\right)^2 \left(3 x^3 + x^2 y + y \left(-1 + y^2\right) + 3 x \left(1 + y^2\right)\right)^2\right)$$

Out[106]=

$$\Big\{\Big(2 x^{11} - x^{10} y - y \left(-1 + y^2\right)^4 \left(1 + y^2\right) + 2 x^9 \left(-13 + 5 y^2\right) + x^8 \left(19 y - 5 y^3\right) + 2 x \left(-1 + y^2\right)^3 \left(1 + 6 y^2 + y^4\right) - 2 x^6 y \left(33 - 30 y^2 + 5 y^4\right) - x^2 y \left(-1 + y^2\right)^2 \left(-19 - 18 y^2 + 5 y^4\right) + 4 x^7 \left(-7 - 18 y^2 + 5 y^4\right) - 2 x^4 y \left(33 + 43 y^2 - 33 y^4 + 5 y^6\right) + 4 x^5 \left(7 - 45 y^2 - 15 y^4 + 5 y^6\right) + 2 x^3 \left(13 + 76 y^2 - 90 y^4 - 4 y^6 + 5 y^8\right)\Big) \Big/ \left(x \left(-1 + x^2 + y^2\right) \left(-1 + x^4 + 12 x y + 2 x^2 y^2 + y^4\right) \left(3 x^3 + x^2 y + y \left(-1 + y^2\right) + 3 x \left(1 + y^2\right)\right)\right),$$

$$-\Big(\Big(x^8 + 4 x^6 \left(-9 + y^2\right) + \left(-1 + y^2\right)^2 \left(1 + 6 y^2 + y^4\right) + x^4 \left(-74 - 68 y^2 + 6 y^4\right) + 4 x^2 \left(-9 - 57 y^2 - 7 y^4 + y^6\right)\Big) \Big/ \left(\left(-1 + x^4 + 12 x y + 2 x^2 y^2 + y^4\right) \left(3 x^3 + x^2 y + y \left(-1 + y^2\right) + 3 x \left(1 + y^2\right)\right)\right)\Big),$$

$$\Big(4 \Big(2 x^{15} - x^{14} y + 2 x^{13} \left(-15 + 7 y^2\right) + x^{12} \left(y - 7 y^3\right) + 2 x^{11} \left(-131 - 96 y^2 + 21 y^4\right) + x^{10} \left(-141 y + 2 y^3 - 21 y^5\right) - y \left(-1 + y^2\right)^4 \left(1 + 7 y^2 + 7 y^4 + y^6\right) - 5 x^8 y \left(87 + 81 y^2 + y^4 + 7 y^6\right) - x^2 y \left(-1 + y^2\right)^3 \left(1 + 21 y^2 + 35 y^4 + 7 y^6\right) + x^9 \left(-230 - 546 y^2 - 510 y^4 + 70 y^6\right) + 2 x \left(-1 + y^2\right)^3 \left(1 - 18 y^2 - 22 y^4 - 18 y^6 + y^8\right) - x^6 y \left(435 + 1460 y^2 + 354 y^4 + 20 y^6 + 35 y^8\right) + x^7 \left(230 + 496 y^2 + 4 y^4 - 720 y^6 + 70 y^8\right) - x^4 y \left(141 + 1025 y^2 + 1050 y^4 + 42 y^6 + 25 y^8 + 21 y^{10}\right) + x^5 \left(262 - 726 y^2 + 516 y^4 + 668 y^6 - 570 y^8 + 42 y^{10}\right) + 2 x^3 \left(15 - 120 y^2 + 105 y^4 - 112 y^6 + 225 y^8 - 120 y^{10} + 7 y^{12}\right)\Big)\Big) \Big/ \left(\left(-1 + x^4 + 12 x y + 2 x^2 y^2 + y^4\right)^2 \left(3 x^3 + x^2 y + y \left(-1 + y^2\right) + 3 x \left(1 + y^2\right)\right)^2\right)\Big\}$$

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