

The Sensible Calculus Program
(draft: work in progress. 7-10-2014)

[With examples using [SAGE](#)  and [GeoGebra](#) - not yet incorporated in the on-line text.
Due to problems with java, sections that included dynamic figures are in revision. 3-22-2017]

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- I. [Preface](#)
- II. [Introduction to the Sensible Calculus Program](#)
- III. [The Agenda and Vision](#)
- IV. [Sample Sections for Previewing](#)
- V. [The Sensible Calculus Program Approach](#)
 - A. [Visualizations and Mapping Diagrams](#)
 - B. [Differential Equations, Tangent Fields, and Euler's Method.](#)
 - C. [Models](#)
 - D. [Taylor Theory](#)
 - E. [Probability](#)
 - F. [Technology](#)
- VI. [A Partial List of Features](#)
- VII. [Bibliography](#)

Preface and Background: A calculus textbook in the twenty-first century is still a very valuable tool for learning the calculus. Understandably, for more than fifty years issues of calculus curriculum and pedagogy have been centered on these books. In the 1970's the MAA Committee on Undergraduate Programs in Mathematics (CUPM) subpanel on calculus gave an excellent analysis of calculus programs suggesting some reasonable remedies to the then current problems. [[MAA.1](#)]. In the 1980's personal computers and graphing calculators suggested a new direction, a technological fix, to some of problems. At the same time a more general educational reform movement proposed other changes such as "writing across the curriculum," problem solving, and an emphasis on relevance and interdisciplinary applications for all levels of mathematics instruction. A call for calculus reform emerged as a national concern supported by the National Science Foundation (NSF). Discussion of problems and proposed solutions appeared in numerous publications such as [[MAA.2](#) and [3](#)] and the newsletter UME TRENDS. As the period of ten years of NSF sponsored calculus reform projects closed in the mid '90's, a summary of these efforts, emphasizing many common features, was presented in [[MAA 6](#), and [7](#)]. The many changes in vision promoted by the period of reform continue today and are still an active area of development. This activity is evident in the continued work on the restatement of goals and directions for undergraduate mathematics programs by the MAA CUPM.

Introduction to the Sensible Calculus Program (SCP): The Sensible Calculus Program reflects work of over thirty years by Professor Martin Flashman that started in **developing a thematic approach to the first year of calculus.** (See [[MAA.4](#)].) Recognizing that a reform of the college precalculus function/algebra course would yield extra benefits in the success of calculus students, in about 1990 Professor Flashman expanded the vision of the program to include calculus preparation. The goal of the current project is the creation of a precalculus and a calculus textbook and a full calculus program with a consistent thematic and conceptual approach. The program will prepare students in the precalculus component for learning calculus by providing motivation, background, concept development, and opportunities for real learning. The calculus component will build on these foundations with an approach that extends precalculus maturation into calculus. Two (among several) distinctive features of the SCP approach will apply to both the precalculus and calculus components:

1. The extensive use of [mapping diagrams \(mapping figures\)](#) to visualize functions along with the traditional use of graphs, and
2. The early and consistent treatment of [modelling contexts](#) with a meaningful connection to issues of [continuous probability](#) that explore the power of mathematics.

These features as related to *The Sensible Calculus Book* will be discussed more thoroughly in later sections. A [partial list of features](#) in the Calculus Component of the SCP is also included at the end of this description.

The Agenda and Vision. Calculus projects over the passed twenty five years have tried to improve the quality of calculus instruction with approaches mixing technology, pedagogical tactics, and curriculum revisions. As a result there has been much progress in rethinking the essential elements for a successful calculus program, leading to a general acceptance of many features of the reformed calculus agenda. Here is a brief statement of the current agenda and vision that the SCP addresses and which provide the fundamental direction for *The Sensible Calculus Book*.

Vision for The Sensible Calculus Program

1. Define clearly and follow organizing concepts and principles for the content of the course;
2. Provide a consistent distinction between motivation, interpretation, and the mathematical concepts of calculus;
3. Connect the numerical, visual, and symbolic aspects of the calculus (called by some the "rule of three");
4. Encourage students to communicate as they solve problems (the "fourth" rule);
5. Use current technology to enhance concepts, not merely to facilitate mechanical approaches to computation, graphing, or symbolic manipulation;
6. Use applications, conceptual explorations, and history to weave together the many strands of mathematics;
7. Provide opportunities for students to control their own learning and understanding and to be creative while expressing their own views; and
8. Improve student backgrounds and motivation before calculus.

Sample sections, draft versions of sections now available on-line: 5-28-2014
Warning: Due to problems with java, some of these sections are in revision. [February, 2017]
[Available for Review with permission only- all rights reserved. flashman@humboldt.edu]

Chapter 0: Introduction, Background, and Other Goodies 0.A What is Calculus? 0.B.1. Numbers, Units, Coordinates, and Variables. 0.B.2 Functions-Introduction and Review.	0.B.3. [More than Just] Lines. 0.C Models and Mathematics (Draft) 0.2 Excursion into Lagrange Interpolation.6-28-00
I. Introduction to the Derivative. I.A [Motivation] Finding the Slope of a Tangent Line I.B [Motivation] Estimating Instantaneous Velocity (Draft revised 2/1/2014)	I.E.3 Derivatives of the sine and cosine functions. I.G. Using the Derivative - Developing Intuitions. (Draft) I. H. Limits in More Detail.

I.C.1 Probability Distributions and Density. 2-22-03 I.C.2 Economics and Marginal Analysis 10-10-06 I.D. The Derivative of a Function [Concept Definition] I.E Notation -The good, the bad, and the confusing I.F Finding the Derivative of Some Key Functions I.F.2 Derivatives of exponential and logarithmic functions (in part)	I.1.1 The Derivative and Continuity. I.1.2. Intermediate Values I.1.3. The Extreme Value Theorem I.Appendix.1. Alternative Views of Continuity, Sequences and Approximations. I.Appendix.2. Proof of the Intermediate Value Theorem. I.Appendix.3. Proof of the Extreme Value Theorem.
II.The Calculus of Derivatives. II.A The Product And Quotient Rules A.2. Application to Trigonometric Functions II.B The Chain Rule	II.C Derivatives for Implicit Functions D Applications: More Related Rates .Rational Powers. E (optional) Inverse Functions and Derivatives. F. Higher Order Derivatives.
III.Applications of the Derivative. III.A.1 (html 10/19/2014) (old pdf) Approximations Using the Derivative.(The differential) III.A.2 (html 10/9/2014) (pdf)Approximations Using the Derivative.(Newton's Method) III.B.1 (html 11/12/2014) III.B.1 Extrema.(Monotonic Functions) III.B.2 (html 11/12/2014) III.B.2 Extrema (Local Extremes & MVT)	III.C Graphic Features of the 2nd Derivative (draft) III.D Graphing in General. E Another Look at Applications to Physics , Probability, and Economics.
IV. Introduction to Differential Equations and the Definite Integral. IV Frames Version of Chapter 4 IV.A Differential Equations from an Elementary Viewpoint IV.B. Solving Differential Equations: Beginning Theory and Calculus. IV.C. Motion and Differential Equations. IV.D Tangent Fields and Integral Curves	IV.E Euler's Method IV.F Euler's Method Meets The Position, Cost, and Area Problems IV.G Differential Equations and Area IV. IV.H The Fundamental Theorem of Calculus: A Differential Equations Approach IV.H Appendix: Proof of FT of C II [$P(x) > 0.$] 6-9-02
V. The Definite Integral. V.A The Definite Integral (draft) 3-14-03 V.B Key Properties of the definite integral V.C The Fundamental Theorem of Calculus.(draft) 11-25-14 V.D Numerical Integration- (in part)	V.E Riemann Sums and Definition of the Definite Integral. F. Remarks on Liebniz, the Differential, and Infinitesimals. V.G Applications of the Definite Integral -Two Views. H. The Calculus of the Definite Integral.
VI. Model Related Differential Equations VI.A Differential Equations & Models- The Exponential Function VI.B Differential Equations & Models- The Natural Logarithm Function	VI.C Connecting the Natural Logarithm and Exponential Functions VI.D More Models & Inverse Trigonometry
VII. Integration, Differential Equations, and the Trigonometric Functions. A. Summary of Elementary Integration Results. B. Applications of Substitution. VII.C. Integration by Parts. -Reduction Formulae. VII.D. Integration of Trigonometric Functions.(2014 draft)	VII.E Trigonometric Substitutions. VII.F Integration of Rational Functions G. Integration by Table. VII.H. Separation of Variables. I. Second Order Differential Equations.
VIII. More Applications of the Definite Integral. A. Refresher on Riemann Sums, Differential Equations and the Definite Integral. VIII.B Arc Length C. Volumes. D. Improper Integrals: Extending the Concepts of Integration.	E. Physics and Engineering Applications. F. Probability and Calculus (Darts : Mean of a Random Variable) (The Normal Distribution : an important integral.) G. Economics and Calculus.
IX. Differential Equations and Polynomials: Taylor's Theorem. IX.A Taylor Theory for e^x IX.B MacLaurin Polynomials and Taylor	IX.C MacLaurin Polynomials: How to Find Them (Draft) IX.D Taylor Polynomials (Draft)
X. Sequences and Series: Fundamental Properties. X.A Simple Examples and Definitions: Visualizing Sequences. (Draft) X.B.1-4 Series: Some Key Examples.(Draft in part)	X.B.5 Tests for Divergence and Convergence. (Draft in Part)
XI. Power Series: Polynomials and Series. XI.A Power Series (Draft)	

The Calculus Text

The Sensible Calculus Program Approach: The SCP focuses on the themes of differential equations [[FL](#)] and estimation throughout the first year of calculus, using modelling as a central motivation for applications of the calculus [[FL2](#)]. In the subsequent third semester or fourth quarter study of the calculus of several variables and the vector calculus these themes continue making an altogether unified and coherent approach to calculus. These themes and motivation have gained general approval from many of today's best selling textbooks. Unfortunately, there is no book today that succeeds in presenting the calculus consistently with these underlying principles. Along with thematic consistency, an important and unique pedagogical feature of the SCP text is the consistent use of interpretations to provide meaning for calculus concepts. Frequently examples of models or arguments will be presented before more general applications and proofs. This organization is based on the pedagogical principle that students can understand the specific and particular in experience and then generalize more easily then they can understand a general proposition or proof and the apply it to the particular.

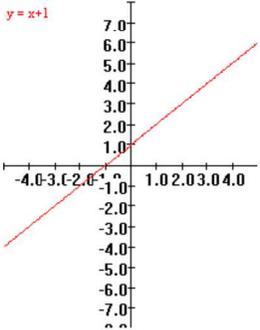
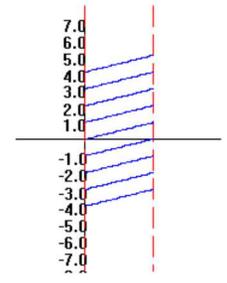
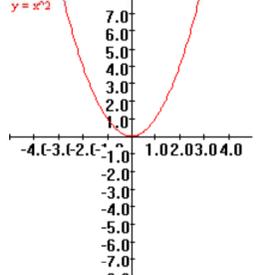
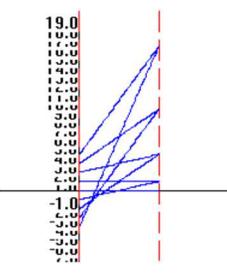
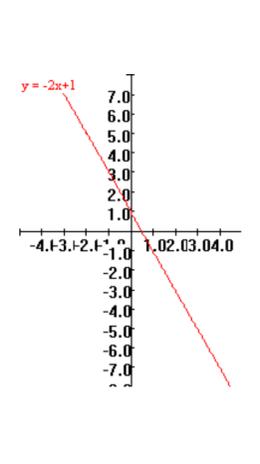
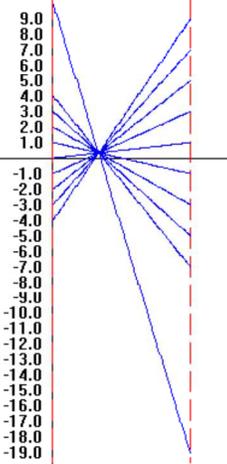
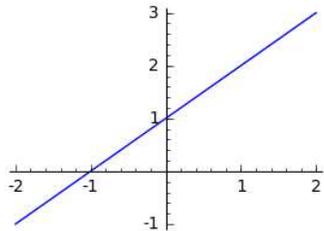
Visualizations and Mapping Diagrams: Practically all major calculus concepts and techniques are interpreted with at least two views:

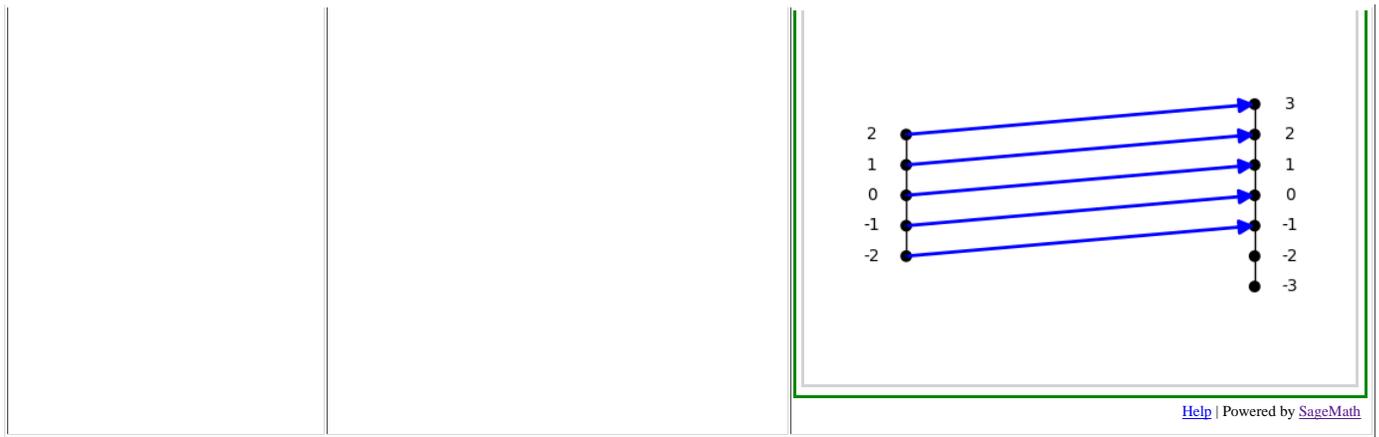
1. A **geometric view** using **cartesian graphs** to visualize functions, and
2. A **dynamic/motion view** using **mapping diagrams (mapping figures)** to visualize functions. (See Figure 1 for some examples of mapping diagrams and corresponding graphs.)

A mapping diagram visually distinguishes the argument of a function from its value by representing each as a point on distinct parallel source (a on the X-Axis) and target lines (b on the Y- axis). These points are then connected with a line segment or an arrow joining a and b . This visualization is in contrast to the traditional Cartesian graph that consolidates the argument and function value visually at a single point (a, b) , located at the intersection of lines, $X = a$ and $Y = b$.

Many students in their first calculus course are still developing their basis for understanding the function concept. The combination of mapping diagrams with Cartesian graphs gives a dual approach to interpreting and visualizing functions that has been particularly helpful for these students. It also is a great aid in making sense of many calculus concepts and results, such as the chain rule and the differential. To make effective use of these visual tools the SCP employs them not merely in one or two instances as some current texts, but regularly invokes them in a variety of contexts [FL1]. This is central to the visually and conceptually balanced approach of the SCP. [See also [FL5], [S&J], and [G] for more on mapping diagrams.]

Figure 1. Examples of Cartesian Graphs and Mapping Diagrams with Tables.

<table border="1"> <thead> <tr> <th>x</th> <th>$f(x) = x + 1$</th> </tr> </thead> <tbody> <tr> <td>2</td> <td>3</td> </tr> <tr> <td>1</td> <td>2</td> </tr> <tr> <td>0</td> <td>1</td> </tr> <tr> <td>-1</td> <td>0</td> </tr> <tr> <td>-2</td> <td>-1</td> </tr> </tbody> </table>	x	$f(x) = x + 1$	2	3	1	2	0	1	-1	0	-2	-1		
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<p>Using SAGE  sage to do this: Click on the Evaluate button. You can change the function definition for the graph and also adjust the axes for the mapping diagram.</p>	<div style="border: 1px solid gray; padding: 5px;"> <p style="text-align: center;">Evaluate</p> <div style="border: 1px solid green; padding: 5px; margin: 5px;"> <p>f <input style="width: 100px;" type="text" value="x + 1"/></p>  </div> <p style="text-align: right; font-size: small;">Help Powered by SageMath</p> </div>	<div style="border: 1px solid gray; padding: 5px;"> <p style="text-align: center;">Evaluate</p> <div style="border: 1px solid green; padding: 5px; margin: 5px;"> <p>f <input style="width: 100px;" type="text" value="x + 1"/></p> <p>inputs <input style="width: 50px;" type="text" value="-2"/> <input style="width: 50px;" type="text" value="2"/> (-2, 2)</p> <p>outputs <input style="width: 50px;" type="text" value="-3"/> <input style="width: 50px;" type="text" value="3"/> (-3, 3)</p> </div> </div>												



Using [GeoGebra](#) to do this:

You can move the point on the x axis in the mapping diagram see the function values vary with the indicated value of x.

This point also moves the related point on the graph.

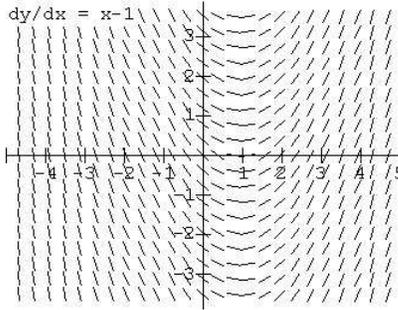
You can change the function definition for the graph and the mapping diagram.



Martin Flashman, 26 June 2013, Created with [GeoGebra](#)

Differential Equations, Tangent Fields, and Euler's Method. The use of tangent (slope or direction) fields (see [Figure 2](#)) to visualize differential equations and Euler's method for estimating solutions to initial value problems (see [Figure 3](#)) are elements of the calculus curriculum that have won general acceptance in that past ten years as evidenced by their inclusion the current Advanced Placement syllabus and the best selling texts. Professor Flashman recognized the importance of these tools for understanding the calculus early in his development of The Sensible Calculus Program and has worked over twenty years with students to find presentations for these tools that are direct and easy to follow. As a result of all this experience, *The Sensible Calculus Book* combines an early and extensive discussion of differential equations with a visual approach through tangent fields [\[FL2\]](#) and a numerical approach using Euler's method [\[FL1\]](#). Throughout the text the combined numeric and visual approaches complement conventional symbolic discussions in topics such as solving indefinite integrals, motivating the definite integral and the Fundamental Theorem of Calculus, finding precise function solutions to initial value problems, and even more advanced symbolic techniques such as the separating variables. The dynamic features of [SAGE](#)  and [GeoGebra](#) are being incorporated as sections are being revised (2014-?) to bring these tools directly to users.

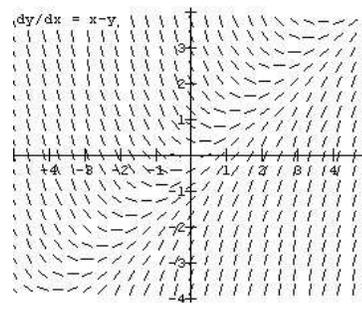
Figure 2: Examples of tangent fields for differential equations.



Tangent Field for $y' = x - 1$.

Using [SAGE](#) to do this: Click on the Evaluate button.
You can change the function for the Tangent Field

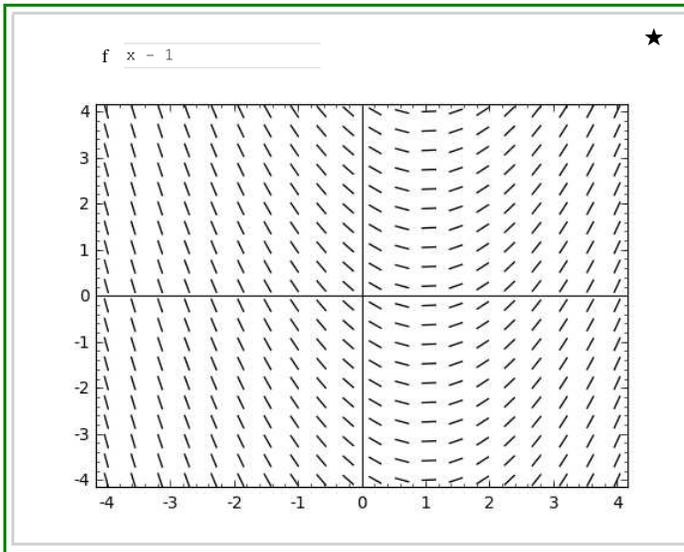
Evaluate



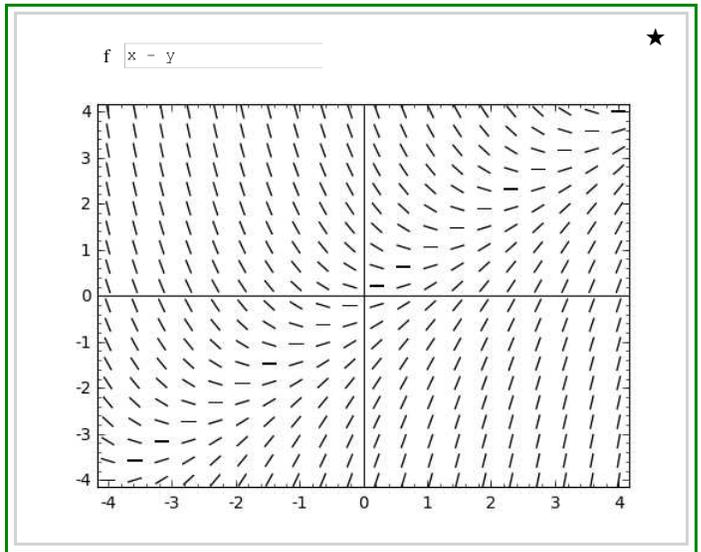
Tangent Field for $y' = x - y$.

Using [SAGE](#) to do this: Click on the Evaluate button.
You can change the function for the Tangent Field

Evaluate



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Figure 3. Example of Table for Euler's Method

Estimating $f(1)$ from $y' = P(x, y) = 2x - y$ with $f(0) = 2, N = 4, x_N = 1, dx = h = \frac{1}{4}$.

x	y [approximate using $f(x + dx) = f(x) + dy$]	$y' = P(x, y) = 2x - y$	$dy = y' dx$
0	2 [given, = $f(0)$.]	$2(0) - 2 = -2$	$(-2)(1/4) = -1/2$
1/4	$2 + (-1/2) = 3/2$	$2(1/4) - 3/2 = -5/4$	$(-5/4)(1/4) = -1/4$
1/2	$3/2 + (-1/4) = 5/4$	$2(1/2) - 5/4 = -1/4$	$(-1/4)(1/4) = -1/16$
3/4	$5/4 + (-1/16) = 19/16$	$2(3/4) - 19/16 = 5/16$	$(5/16)(1/4) = 5/64$
1	$19/16 + 5/64 = 81/64$		

Using [GeoGebra](#) to do this: Estimating $f(1)$ from $y' = P(x, y) = 2x - y$ with $f(0) = 2, N = 4, x_N = 1, dx = h = \frac{1}{4}$.

You can change the values of the end points on the axis using the sliders or enter them in the boxes in the mapping diagram.

Use the slider to change the value of N up to 100.

You can change the function definition for the graph and the mapping diagram by entering a new function in the box.

You can change the value of $f(a)$ directly in the spreadsheet cell.

Click on the checkbox to show a slopefield with resolution also controlled by N

1	x	f(x)	f'(x)	df=f'(x)dx
2	0	2	-2	-0.5
3	0.25	1.5	-1	-0.25
4	0.5	1.25	-0.25	-0.0625
5	0.75	1.1875	0.3125	0.0781
6	1	1.2656	0.7344	0.1836
7	1.25	1.4492	1.0508	0.2627
8	1.5	1.7119	1.2881	0.322
9	1.75	2.0339	1.4661	0.3665
10	2	2.4005	1.5995	0.3999
11	2.25	2.8003	1.6997	0.4249
12	2.5	3.2253	1.7747	0.4437
13	2.75	3.6689	1.8311	0.4578
14	3	4.1267	1.8733	0.4683
15	3.25	4.595	1.905	0.4762
16	3.5	5.0713	1.9287	0.4822
17	3.75	5.5535	1.9465	0.4866
18	4	6.0401	1.9599	0.49
19	4.25	6.5301	1.9699	0.4925
20	4.5	7.0226	1.9774	0.4944

Martin Flashman, 2 July 2014, Created with [GeoGebra](#)

Another example using [GeoGebra](#) with $y' = P(x)$ as a function of x alone:

Estimating $f(2)$ from $y' = P(x) = 2x + 1$ with $f(0) = 0$, $N = 10$, $x_N = 2$, $dx = h = \frac{1}{5} = 0.2$.

You can change the values of the end points on the axis using the sliders or enter them in the boxes in the mapping diagram.

Use the slider to change the value of N up to 100.

You can change the function definition for the graph and the mapping diagram by entering a new function in the box.

You can change the value of $f(a)$ directly in the spreadsheet cell.

9999	A	B	C	D	999
1	x	f(x)	f'(x)	df=f'(x)dx	
2	0	0	1	0.2	
3	0.2	0.2	1.4	0.28	
4	0.4	0.48	1.8	0.36	
5	0.6	0.84	2.2	0.44	
6	0.8	1.28	2.6	0.52	
7	1	1.8	3	0.6	
8	1.2	2.4	3.4	0.68	
9	1.4	3.08	3.8	0.76	
10	1.6	3.84	4.2	0.84	
11	1.8	4.68	4.6	0.92	
12	2	5.6	5	1	
13	2.2	6.6	5.4	1.08	
14	2.4	7.68	5.8	1.16	
15	2.6	8.84	6.2	1.24	
16	2.8	10.08	6.6	1.32	
17	3	11.4	7	1.4	
18	3.2	12.8	7.4	1.48	
19	3.4	14.28	7.8	1.56	

Martin Flashman, 10 June 2014, Created with [GeoGebra](#)

Using [SAGE](#)  to do this: Click on the Evaluate button.

You can change the derivative, $P(x, y)$, the initial values, $f(a) = b$, the differential, $dx = h$, and $c = x_N$, to estimate $f(c)$ using Euler's Method.

Evaluate

P $2*x - y$ ★

a

b	2		
h	1/4		
c	1		
x	Y	h*f(x,y)	
0	2	-1/2	
1/4	3/2	-1/4	
1/2	5/4	-1/16	
3/4	19/16	5/64	
1	81/64	47/256	

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Models: Connecting applications of the calculus to the real world is not new. For centuries the calculus has been used to understand and plan in science and engineering. In traditional calculus texts, geometric and physical applications appear as motivation primarily for the calculus concept of the derivative. Otherwise applications usually appear only after a technique or concept has been presented. *The Sensible Calculus Book* presents applications as a part of the modeling process in many contexts, both traditional and more contemporary, especially in economics and continuous probability. These contexts are threaded throughout the text, from the initial chapters on functions and the motivation of the derivative, to the first sections on differential equations and the introduction of both the indefinite and definite integrals, to the understanding of applications of the definite integral and the use of Taylor theory and concepts of infinite series.

Frequently a model is presented for motivation and exploration before the calculus concept or technique as in the treatment of transcendental functions with [differential equations](#). For example, a heuristic model for learning introduces a differential equation context. The basic premise is that the rate of learning is often a positive yet decreasing function of time, for instance i) $L'(t) = \frac{1}{t}$ with $t > 0$ or ii) $L'(t) = \frac{1}{t^2+1}$. For the first equation, after a consideration of units and time scales, the boundary condition $L(1) = 0$ is established. Questions such as what will be the short and long run behavior of the learning function are investigated using tangent fields and Euler's method, as well as the derivative form of the Fundamental Theorem of Calculus. All this leads to a treatment of the natural logarithm function in the context of a differential equations model for learning.

Likewise population (exponential growth and predator-prey) models motivate the exponential and trigonometric functions. All of these functions are treated with their traditional definitions as well. The object of the modeling approach is not to give formal rigorous treatments of these functions, but to show how the common functions arise and/or remain significant because they solve modelling problems expressed with differential equations. Second order differential equations and population models motivate the hyperbolic functions as student investigations.

Taylor Theory: Polynomial functions and geometric series have been a part of the calculus since the earliest work of Newton and Leibniz that has developed into what is now described as "Taylor theory." *The Sensible Calculus Book* approaches Taylor theory as a distinct and important part of calculus, not a mere appendage at the end of a year's course treatment of infinite series. The text's development is based on the unifying themes of differential equations and estimation. The calculus of Taylor polynomials (not series) appears as a tool for approximating difficult definite integrals such as $\int_0^1 e^{-x^2} dx$ with a sensible control on the error.

Estimating the solution to differential equations such as $y'' = -y$ or $y'' = y$ with $y(0) = 1$ and $y'(0) = 1$ [which characterize the cosine and the hyperbolic cosine functions] provides additional motivation for the convergence questions of infinite series. Here again the problems we want to solve are placed in the text before the techniques or theory.

Infinite sequences and series analysis discuss Taylor theory examples from the beginning along with the traditional examples of geometric and harmonic series. Historical connections to Newton's work in estimating the values of the natural logarithm illustrate how geometric series played a significant role in the showing the power of the early calculus for computation and estimation.

Probability: One of the most important applications of the calculus in current science and engineering is its use as a foundation for understanding probability and statistics. The inclusion of a section or two on this application in the best selling texts demonstrates the increased awareness by authors and teachers of the importance of this application. *The Sensible Calculus Book* does more than include a token section on probability and calculus. It includes continuous probability in the development of concepts of both differential and integral calculus.

Starting with a simple experiment of throwing a dart at a unit circle in the background pre-calculus sections, the text develops the distance from where the dart lands to the center of the circle as a random variable for further investigation. Trying to understand the distribution function for this random variable eventually leads to its density function, i.e., its derivative. The text treats probability concepts frequently as interpretations of the calculus. The evaluation form of the Fundamental Theorem of Calculus shows the integral relation of the distribution function to the density function. Other probability concepts, such as the median, mode, and mean are introduced with an appropriate calculus concept.

The Sensible Calculus' unique approach to probability follows the philosophies of both the NCTM Standards and the California Framework that call for the exploration of probability concepts as a regularly encountered strand in the fabric of mathematics. Whether a student is planning to be an engineer, a physician, or some kind of scientist, one of the most useful parts of their mathematics training will be probability and statistics. Making probability more relevant throughout the course allows students to see the interaction of calculus with this universal application. It enriches their understanding of both subjects as well as showing the application of continuous mathematics to nondeterministic situations.

Technology: Though the SCP is not committed to any specific technology, it presumes access to technology capable of graphing and computation as well as some programming features in the event that computation or visual features are not pre-designed in the technology. The text is not driven by technology, but it does presume (through its problem sets) that students and teachers will use technology to explore and enhance concepts as well as to solve problems.

- Use of the world wide web including [java graphics](#) and downloadable [Microsoft Excel spreadsheets](#) will make many text discussions more interactive. Use of [SAGE](#) and [Geogebra](#) will be incorporated when appropriate to the on-line version.[6-28-2013]
- Even older graphing calculators such as the TI 81 or the Casio 7700 should be adequate.
- Something comparable to the HP48-G, HP 39, HP 49 or the TI-89, TI-92 or TI - Inspire is preferable for the ease and power they can bring to calculus.
- For computer software, use of [Winplot](#) (dynamic graphing freeware from Peanut Software) will be supported as an easy learning tool for most of the concepts.

- A Computer Algebra System (CAS), such as *Mathematica*, DERIVE, MATLAB, or MAPLE, will not be presumed, but should be very useful to those who have access to such a system. Eventually study supports for those using a CAS will be developed and either published as supplements or made available through the internet.

Partial List of Features in the Calculus Component of the SCP

Pedagogy:

- Consistent and balanced use of **mapping (mapping) figures with cartesian graphs** to visualize functions and calculus concepts.
- **Learning builds from experience** with examples to generalization and abstraction.
- Meaningful use of **history** to introduce and develop concepts.
- Technology used to enhance exploration of concepts with rule of three.
- Different **assignment formats**: team assignments, journals, outside reading, research assignments, oral presentations (not yet incorporated in the text).
- **Synthesizing applications** that put together many tools to explore a particular application in detail. [The integral of $\exp(-x^2)$]

Differential Equations and Integration:

- **Early tangent fields and integral curves** to visualize differential equations and solutions.

Example: Draft of Section on [Tangent Fields](#).

- **Euler's method** for numerical estimation of solutions for differential equations visualized with tangent fields and mapping diagrams.

Example: Draft of Section on [Euler's method](#).

- Differential equation and area **motivation for the definite integral**.
- Methods of integration
 - connected to applications and modelling problems
 - visualized using tangent fields
- **Model motivation (DE's) for transcendental functions** (exp, ln, trig, and inverse trig)

Examples: Draft of Section [VI.A](#) connecting differential equations to the exponential function and Section [VI.B](#) connecting differential equations to the natural logarithm function. Draft of [VI.C](#) connecting the natural logarithm and exponential functions.

Taylor theory and Series:

- **Taylor Theory** applications without reliance on infinite series.

Example: Draft of Section [IX.A](#) introducing Taylor Theory for e^x with applications.

- Analysis of infinite sequences and series explains prior calculus experience.
- **Series focus on applications** to themes of estimating functions, definite integrals and solution to differential equations.

Example: Draft of part of Section [X.B](#) Series: Some Key Examples.

Probability:

- Probability as a consistent interpretation for the calculus.
- The distribution of a continuous random variable as the primitive concept for probability.
- The density function presented as the derivative of the distribution function.
- Median and mode related to calculus concepts.
- The Fundamental Theorems of Calculus interpreted with density and distribution functions.
- The mean of a random variable introduced without weighted averages.
- Expected value and variation.
- Normal Curve parameters
- Probability and infinite series.

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